A regional segregation problem as the vanishing latent heat of a sequence of stochastic Stefan-like problem

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Abstract

We study an approximation of the regional segregation problem of two competing species which is modeled by a two-components reaction-diffusion system. More precisely, we shall prove the convergence of the sequence of solutions to Stefan-type problems to the solution of the regional segregation problem, when we make the latent heat converges to zero.

Keywords: regional segregation problem, stochastic porous media, Wiener process, Stefan problem

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1 Introduction

In the present work, we are interested in the study of an approximation of the regional segregation problem of two competing species which is modeled by the following two-components reaction-diffusion system:

$$
\left\{
\begin{array}{ll}
du\,(t) = \left(d_1 \Delta u\,(t) + h\,(u\,(t))\right) dt + B\,(u)\,dW_t, & t > 0,\ x \in \mathcal{O}_u\,(t)\,, \\
dv\,(t) = \left(d_2 \Delta v\,(t) + h\,(v\,(t))\right) dt + B\,(v)\,dW_t, & t > 0,\ x \in \mathcal{O}_v\,(t)\,, \\
u = v = 0, & t > 0,\ x \in \Gamma\,(t)\,, \\
d_1 \dfrac{\partial u}{\partial n} = -d_2 \dfrac{\partial v}{\partial n}, & t > 0,\ x \in \Gamma\,(t)\,, \\
u = v = 0, & t > 0,\ x \in \partial\mathcal{O}, \\
u\,(x,0) = u_0\,(x)\,, & x \in \mathcal{O}_u\,(0)\,, \\
v\,(x,0) = v_0\,(x)\,, & x \in \mathcal{O}_v\,(0)\,, \\
\Gamma\,(0) = \Gamma_0, &
\end{array}
\right. \tag{1}
$$

where $\Gamma(t)$ is the interface which separates $\mathcal{O}$ into two subregions

$$\mathcal{O}_u(t) = \{x \in \mathcal{O} \mid u(x) > 0 \text{ and } v \equiv 0\}$$

and

$$\mathcal{O}_v(t) = \{x \in \mathcal{O} \mid v(x) > 0 \text{ and } u \equiv 0\}$$

corresponding to the two competing species. We denoted by d_1 and d_2 the diffusion rates, by r_1 and r_2 the intrinsic growth rate, and by k_1 and k_2 the carrying capacity of u and v, respectively. For more details see [7].

The previous equation can be rewritten in this form, which is more appropriated to be translated into a "porous media" formulation.

$$(2) \qquad \begin{cases} du(t) = (d_1 \Delta u(t) + h(u(t)))\, dt + B(u)\, dW_t, & \{u > 0\} \cap Q_T, \\ du(t) = (d_2 \Delta u(t) + h(u(t)))\, dt + B(u)\, dW_t, & \{u < 0\} \cap Q_T, \\ u = 0, & t > 0,\ x \in \Gamma(t), \\ d_1 \dfrac{\partial u^+}{\partial n} = d_2 \dfrac{\partial u^-}{\partial n}, & t > 0,\ x \in \Gamma(t), \\ u = 0, & t > 0,\ x \in \partial\mathcal{O}, \\ u(x,0) = u_0(x), & x \in \mathcal{O}, \\ \Gamma(0) = \Gamma_0. \end{cases}$$

Since the boundary between the two phases is difficult to describe, we shall study this equation as the limit of a sequence of two-phase Stefan type problems. More precisely, we shall prove the convergence of the sequence of solutions to Stefan-type problems to the solution of the regional segregation problem, when we make the latent heat converges to zero.

A similar problem of vanishing latent heat in a sequence of two-phase Stefan problem was studied by Tarzia (see [11], [12] and [13]) in a deterministic case without reaction terms. For the deterministic case with particular forms of reaction diffusion terms see [8]. The stochastic case with linear multiplicative noise can be seen as a particular case of [6]. To the best of our knowledge, the convergence in the stochastic case for an equation with reaction diffusion term, has not yet been treated before the present paper.

We shall introduce the following two-phase Stefan stochastic differential equation

$$(3) \qquad \begin{cases} du(t) = (d_1 \Delta u(t) + h(u(t)))\, dt + B(u)\, dW_t, & \{u > 0\} \cap Q_T, \\ du(t) = (d_2 \Delta u(t) - h(-u(t)))\, dt - B(-u)\, dW_t, & \{u < 0\} \cap Q_T, \\ u = 0, & t > 0,\ x \in \Gamma(t), \\ \sigma V_n = -d_1 \dfrac{\partial u^+}{\partial n} - d_2 \dfrac{\partial u^-}{\partial n}, & t > 0,\ x \in \Gamma(t), \\ u(t,x) = 0, & t > 0,\ x \in \partial\mathcal{O}, \\ u(0,x) = u_0(x), & x \in \mathcal{O}, \\ \Gamma(0) = \Gamma_0, \end{cases}$$

where $\mathcal{O}$ is a bounded open subset of $\mathbb{R}^d$ with smooth boundary $\partial\mathcal{O}$ and $T > 0$. We set $Q_T = \mathcal{O} \times [0,T] \times \Omega$ and

$$\Gamma(t) = \{(\xi, t, \omega) \in \mathcal{O} \times [0,T] \times \Omega\ ;\ u(\xi, t, \omega) = 0\}.$$

We denote by n the unit normal vector to the free boundary $\Gamma(t)$ and by V_n its normal velocity.

The equation is considered in the by now classical monotonicity setting by considering the Gelfand triple $L^2(\mathcal{O}) \subset \left(H_0^1(\mathcal{O})\right)^* =: H^{-1}(\mathcal{O}) \subset \left(L^2(\mathcal{O})\right)^*$ where $H_0^1(\mathcal{O})$ is the usual Sobolev space corresponding to the Dirichlet boundary conditions, and with norm $|\cdot|_1$, $\left(H_0^1(\mathcal{O})\right)^* = H^{-1}(\mathcal{O})$ is the dual of $H_0^1(\mathcal{O})$ with norm $|\cdot|_{-1}$, and $\left(L^2(\mathcal{O})\right)^*$ is the dual of $L^2(\mathcal{O})$ with respect to the structure of $H^{-1}(\mathcal{O})$. For details, see Example 4.1.11 from [10].

The cylindrical Wiener process W is defined on $L^2(\mathcal{O})$ by setting

$$W_t = \sum_{k=1}^{\infty} \beta_k(t)\, e_k,$$

where $\{\beta_k\}_{k\in\mathbb{N}}$ is a sequence of mutually independent standard one-dimensional Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P})$ satisfying the usual conditions and $\{e_k\}_k$ is an orthonormal basis in $L^2(\mathcal{O})$, formed by the normalized sequence of eigenfunctions of the Laplace operator with Dirichlet boundary conditions.

The positive parameter σ corresponds to the latent heat in the classical Stefan problem.

We shall assume the following Hypotheses.

($\mathbf{H}_1$) The function $h : H^{-1}(\mathcal{O}) \to H^{-1}(\mathcal{O})$ is Lipschitz continuous on $H^{-1}(\mathcal{O})$. Furthermore, h keeps $L^2(\mathcal{O})$ invariant and its restriction to $L^2(\mathcal{O})$ is Lipschitz continuous with the usual norm $|\cdot|_2$.

($\mathbf{H}_2$) For the stochastic noise we shall define the operator

$$B : H^{-1}(\mathcal{O}) \to \mathcal{L}_2\left(L^2(\mathcal{O}); H^{-1}(\mathcal{O})\right),$$

where $\mathcal{L}_2\left(L^2(\mathcal{O}); H^{-1}(\mathcal{O})\right)$ is the usual space of Hilbert-Schmidt operators. We shall assume that

$$\begin{aligned} \|B(u) - B(v)\|^2_{\mathcal{L}_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))} &\leq C\,|u-v|^2_{-1}, \\ \|B(u) - B(v)\|^2_{\mathcal{L}_2(L^2(\mathcal{O});L^2(\mathcal{O}))} &\leq C\,|u-v|^2_{2}, \end{aligned}$$

and

$$\begin{aligned} \|B(u)\|^2_{\mathcal{L}_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))} &\leq C\,|u|^2_{-1}, \\ \|B(u)\|^2_{\mathcal{L}_2(L^2(\mathcal{O});L^2(\mathcal{O}))} &\leq C\,|u|^2_{2}, \end{aligned}$$

where C denotes a generic constant which may change from a line to another.

Remark 1 *1. The assumption* ($\mathbf{H}_1$) *is interesting and reasonable because it can be obtained by the following construction. Let us consider* $(e_j)_{j\geq 1}$ *to be the eigen-functions of the Laplace operator with Dirichlet homogeneous conditions belonging to* $L^2(\mathcal{O})$ *with the associated eigen-values* λ_j *ordered non-decreasingly and forming an orthonormal basis in* $L^2(\mathcal{O})$*. Furthermore, we denote by* $\widetilde{e}_j := \sqrt{\lambda_j} e_j$ *in order to obtain an orthonormal basis in* $H^{-1}(\mathcal{O})$*.*

With these notations, a standard construction of such nonlinear coefficients h *consists in picking a family of Lipschitz-continuous real-functions* $\left(h^j\right)_{j\geq 1}$ *whose Lipschitz constants are set to be* $\left[h^j\right]_1$ *and set*

$$h(x) := \sum_{j\geq 1} \gamma_j \left\langle h^j\left(\prod\nolimits_{span\{e_1,e_2,\ldots,e_j\}} x\right), \tilde{e}_j \right\rangle_{-1} \tilde{e}_j, \quad \forall x \in \mathbb{R},$$

for a family $\{\gamma_j : j \geq 1\} \subset \mathbb{R}$ such that $\sum_{j\geq 1} \gamma_j^2 \left[h^j\right]_1^2 \lambda_j < \infty$. The orthogonal project is sought in H^{-1}.

2. *Note that the assumptions above on the operator B from the noise are satisfied if we assume that B is linear or if we construct it in the spirit of the previous assertion.*

2 Existence for the Stefan-type equation

We shall treat equation (3) in the framework of nonlinear multi-valued problems of monotone type. To this purpose we set

$$b_\sigma(r) = \begin{cases} r, & \text{if } r < 0, \\ [0,\sigma], & \text{if } r = 0, \\ r+\sigma, & \text{if } r > 0. \end{cases}$$

$$\mathcal{D}(r) = \begin{cases} d_2 r, & \text{if } r \leq 0, \\ d_1 r, & \text{if } r > 0. \end{cases}$$

We can rewrite the equation (3) as a stochastic variational inequality

$$\begin{cases} db_\sigma(u) \ni (\Delta\mathcal{D}(u) + h(u))\, dt + B(u)\, dW_t, & \mathcal{O} \times (0,T), \\ \mathcal{D}(u) = 0, & \partial\mathcal{O} \times (0,T), \\ b_\sigma(u) = b_\sigma(u_0) = b_\sigma^0, & \mathcal{O} \times \{0\}. \end{cases}$$

By the classical change of variable $b_\sigma(u) = X_\sigma$ we get the porous media type equation

$$\text{(4)} \qquad \begin{cases} dX_\sigma = \left(\Delta\mathcal{D}\left(b_\sigma^{-1}(X_\sigma)\right) + h\left(b_\sigma^{-1}(X_\sigma)\right)\right) dt + B\left(b_\sigma^{-1}(X_\sigma)\right) dW_t, & \mathcal{O} \times (0,T), \\ \mathcal{D}\left(b_\sigma^{-1}(X_\sigma)\right) = 0, & \partial\mathcal{O} \times (0,T), \\ X(0) = b_\sigma^0, & \mathcal{O} \times \{0\}. \end{cases}$$

By an elementary calculus we have

$$b_\sigma^{-1}(r) = \begin{cases} r, & r < 0, \\ 0, & r \in (0,\sigma), \\ r-\sigma, & r > \sigma, \end{cases}$$

and by denoting

$$\mathcal{D}_\sigma(r) = \mathcal{D}\left(b_\sigma^{-1}(r)\right) = \begin{cases} d_2 r, & r < 0, \\ 0, & r \in (0,\sigma), \\ d_1(r-\sigma), & r > \sigma, \end{cases}$$

$$h_\sigma(r) = h\left(b_\sigma^{-1}(r)\right) = \begin{cases} h(r), & r < 0, \\ h(0), & r \in (0,\sigma), \\ h(r-\sigma), & r > \sigma, \end{cases}$$

and

$$B_\sigma(r) = B\left(b_\sigma^{-1}(r)\right),$$

we can rewrite equation (4) as

$$\text{(5)} \qquad \begin{cases} dX_\sigma = (\Delta\mathcal{D}_\sigma(X_\sigma) + h_\sigma(X_\sigma))\, dt + B_\sigma(X_\sigma)\, dW_t, & \mathcal{O} \times (0,T), \\ X_\sigma = 0, & \partial\mathcal{O} \times (0,T), \\ X_\sigma(0) = b_\sigma^0, & \mathcal{O} \times \{0\}. \end{cases}$$

By using the same argument as in [1] page 545, we get that if we have a solution X_σ to equation (5) then we have also a weak solution to the Stefan type problem (3).

Concerning the existence of a solution for equation (5) we shall use the variational type of solution, which is appropriated to the approach which uses the Gelfand triple (For more details see Definition 4.2.1 from [10]).

For readers convenience we recall the definition of this type of solution.

Definition 2 *A continuous $H^{-1}(\mathcal{O})$-valued, $\mathcal{F}_t-$adapted process $(X_\sigma(t))_{t\in[0,T]}$ is called a solution to (5) on $[0,T]$ if for its $dt\otimes\mathbb{P}$-equivalence class $\widehat{X}$ we have $\widehat{X}\in L^2(\mathcal{O}\times(0,T)\times\Omega)$ and $\mathbb{P}$-a.s.*

$$X_\sigma(t) = b_\sigma^0 + \int_0^t \left(\Delta\mathcal{D}_\sigma\left(\overline{X}_\sigma(s)\right) + h_\sigma\left(\overline{X}_\sigma(s)\right)\right) ds + \int_0^t B_\sigma\left(\overline{X}_\sigma(s)\right) dW_s, \tag{6}$$

for $t\in[0,T]$ and $\overline{X}_\sigma$ is any $L^2(\mathcal{O})$-valued progressively measurable $dt\otimes\mathbb{P}$-version of $\widehat{X}$.

Concerning the existence of the solution, the main difficulty for the equation (5) comes from the fact that the operator b_σ^{-1} is not well posed in the space $H^{-1}(\mathcal{O})$ which is the natural space for the porous media type equations. For this reason we can only treat the following "projected equation".

$$\begin{cases} dX_\sigma^N = \left(\Delta\mathcal{D}_\sigma\left(X_\sigma^N\right) + h_\sigma^N\left(X_\sigma^N\right)\right) dt + B_\sigma^N\left(X_\sigma^N\right) dW_t, & \mathcal{O}\times(0,T), \\ X_\sigma^N = 0, & \partial\mathcal{O}\times(0,T), \\ X_\sigma^N(0) = b_\sigma^0, & \mathcal{O}\times\{0\}, \end{cases} \tag{7}$$

where

$$h_\sigma^N(\cdot) := h\left(b_\sigma^{-1}(\Pi^N(\cdot))\right).$$

We considered, as above, the projection

$$\begin{aligned} \Pi^N &: H^{-1}(\mathcal{O}) \to H^{-1}(\mathcal{O}) \\ \Pi^N(u) &= \sum_{k=1}^N \langle u, \tilde{e}_k\rangle_{-1} \tilde{e}_k = \sum_{k=1}^N \langle u, e_k\rangle_2 e_k. \end{aligned}$$

The reader is invited to note that the application above is an orthonormal projection onto $H^{-1}(\mathcal{O})$ and, in particular, $\left|\Pi^N(u)\right|_{-1} \le |u|_{-1}$.

The operator B_σ is approximated by B_σ^N which is constructed in the same way as h_σ^N.

Remark 3 *Let $\sigma>0$ and h which satisfy the assumption above.*

1. *$\mathcal{D}_\sigma$ is $d_1\vee d_2$-Lipschitz as a real-valued function which implies the same Lipschitz constant for $\mathcal{D}_\sigma$ as an $L^2(\mathcal{O})$ operator. In particular, we have a* uniform *Lipschitz behavior with respect to the parameter $\sigma>0$.*
2. *The same kind of properties holds true for b_σ^{-1} as real-valued, hence $L^2(\mathcal{O})$ operator. The Lipschitz constant is 1.*
3. *Since Π^N is an orthogonal projector on $\mathbb{L}^2$ as well, the coefficients $h_\sigma^N(x) := h\left(b_\sigma^{-1}\left(\Pi^N(x)\right)\right)$ is Lipschitz in $L^2(\mathcal{O})$ and the uniform (in $N\ge1$ and $\sigma>0$) Lipschitz constant is the one h has.*

4. *The introduction of* Π^N *aims at equally guaranteeing the* $H^{-1}(\mathcal{O})$*-Lipschitz property of* h_σ^N *uniformly in* $\sigma > 0$ *(although not in* $N \geq 1$*!). To see this, one writes, for* $x, y \in H^{-1}(\mathcal{O})$,

$$\begin{aligned}
\left|h_\sigma^N(x) - h_\sigma^N(y)\right|_{-1} &\leq [h]_1 \left|b_\sigma^{-1}\left(\Pi^N(x)\right) - b_\sigma^{-1}\left(\Pi^N(x)\right)\right|_{-1} \leq [h]_1 \left|b_\sigma^{-1}\left(\Pi^N(x)\right) - b_\sigma^{-1}\left(\Pi^N(y)\right)\right|_2 \\
&\leq [h]_1 \left(\sum_{1\leq k\leq N} \langle x-y, e_k\rangle_2^2\right)^{\frac{1}{2}} = [h]_1 \left(\sum_{1\leq k\leq N} \frac{1}{\lambda_k}\langle x-y, \tilde{e}_k\rangle_2^2\right)^{\frac{1}{2}} \\
&= [h]_1 \left(\sum_{1\leq k\leq N} \frac{1}{\lambda_k}\langle x-y, -\Delta\tilde{e}_k\rangle_{-1}^2\right)^{\frac{1}{2}} = [h]_1 \left(\sum_{1\leq k\leq N} \lambda_k\langle x-y, \tilde{e}_k\rangle_{-1}^2\right)^{\frac{1}{2}} \\
&\leq \sqrt{\lambda_N}\,[h]_1 \left\|\Pi^N(x-y)\right\|_{-1} \leq \sqrt{\lambda_N}\,[h]_1 \left\|(x-y)\right\|_{-1}.
\end{aligned}$$

5. *Similar arguments are valid for* B_σ^N.

In order to study the existence of the solution for the previous equation we consider the operator

$$E_\sigma^N = \Delta\mathcal{D}_\sigma + h_\sigma^N : D(A^\sigma) \subset H^{-1}(\mathcal{O}) \to H^{-1}(\mathcal{O})$$

where

$$\begin{aligned}
D(A) \quad = \quad & \left\{x \in H^{-1}(\mathcal{O}) \cap L^1(\mathcal{O});\right. \\
& \left.\mathcal{D}_\sigma(x) \in H_0^1(\mathcal{O}),\ h_\sigma^N(x) \in L^2(\mathcal{O})\right\}.
\end{aligned}$$

One can easily see that the operators E_σ^N and B_σ^N from equation (7) satisfy the assumptions ...of the operators A and B, respectively, from Theorem 4.2.4 from [10] and consequently the equation above has a unique solution in the sense of the definition mentioned before.

The fact that we replaced the operators h_σ and B_σ by h_σ^N and B_σ^N is not actually changing the nature of our problem because we have anyway an approximation converging to the limit equation (corresponding to $\sigma = 0$) and for the limit equation we can pass also to the limit for $N \to \infty$. So we really treat the announced problem (1), in the sense that we give a result of convergence of solutions to the solution of equation (1).

3 The convergence of $\sigma \to 0$

In this section we shall prove that the solution X_σ^N to equation (7) converges to the solution X^N of the limit equation which corresponds to $\sigma = 0$, for each N fixed.

Keeping in mind that for $\sigma = 0$, we get the function $b_\sigma^{-1}(r) = b_0^{-1}(r) = r$, we can consider the limit equation

$$(8) \qquad \begin{cases} dX^N = \left(\Delta\mathcal{D}\left(X^N\right) + h^N\left(X^N\right)\right)dt + B^N\left(X^N\right)dW_t, & \mathcal{O}\times(0,T), \\ X^N = 0, & \partial\mathcal{O}\times(0,T), \\ X^N(0) = u_0, & \mathcal{O}\times\{0\}, \end{cases}$$

Note that this equation has a unique solution because it satisfies also the assumptions from Theorem 4.2.4 from [10].

Theorem 4 *Under the assumptions above we have that* $X_\sigma^N \to X^N$ *for* $\sigma \to 0$ *in* $C\left([0,T]; L^2\left(\Omega; H^{-1}(\mathcal{O})\right)\right)$.

Proof.

1. In order to get the convergence above, we shall first need to check that the sequence of resolvents of the operators $\mathcal{D}_\sigma$ converges to the resolvent of the operator $\mathcal{D}$ point-wise.

   More precisely, we have that

   $$(1+\lambda\mathcal{D}_\sigma)^{-1}(y) \to (1+\lambda\mathcal{D})^{-1}(y), \quad \forall y \in \mathbb{R}, \tag{9}$$

   for 1 denoting the identity function and for each fixed $\lambda > 0$, as $\sigma \to 0$.

   Indeed, if we consider, for $y \in \mathbb{R}$ fixed, the equations

   $$x_\sigma + \lambda\mathcal{D}_\sigma(x_\sigma) = y,$$

   and

   $$x + \lambda\mathcal{D}(x) = y,$$

   by taking the difference, we get that

   $$x_\sigma - x + \lambda(\mathcal{D}_\sigma(x_\sigma) - \mathcal{D}(x)) = 0.$$

   First, one writes the obvious inequality

   $$|\mathcal{D}_\sigma(x_\sigma) - \mathcal{D}(x)| \le |\mathcal{D}_\sigma(x_\sigma) - \mathcal{D}(x_\sigma)| + |\mathcal{D}(x_\sigma) - \mathcal{D}(x)|.$$

   Second, the form of $\mathcal{D}_\sigma$ yields $\lim_{\sigma\to 0} \sup_{r\in\mathbb{R}} |\mathcal{D}_\sigma(r) - \mathcal{D}(r)| = 0$ and $|\mathcal{D}(x_\sigma) - \mathcal{D}(x)| \le C|x_\sigma - x|$ for some positive constant C, leading to $|x_\sigma - x| \to 0$ for $\sigma \to 0$, which is the desired conclusion.

2. In the spirit of the classical Yosida approximation (up to a strict monotonicity perturbation), let us introduce, for $\lambda > 0$,

   $$\mathcal{D}_\sigma^\lambda(x) := \frac{1}{\lambda}\left(x - (1+\lambda\mathcal{D}_\sigma)^{-1}(x)\right) + \lambda x = \mathcal{D}_\sigma\left((1+\lambda\mathcal{D}_\sigma)^{-1}(x)\right) + \lambda x, \text{ for } x \in \mathbb{R}.$$

   The reader is invited to note that

   $$\mathcal{D}_\sigma^\lambda(x) \longrightarrow \mathcal{D}^\lambda(x), \quad \forall x \in \mathbb{R},$$

   for each λ fixed and by letting the parameter σ vanish i.e. $\sigma \to 0$.

   For this reason, we shall take the Yosida-like approximation for $\mathcal{D}_\sigma$ and $\mathcal{D}$, and the corresponding equations, i.e.

   $$\begin{cases} dX_\sigma^{N,\lambda} = \left(\Delta\mathcal{D}_\sigma^\lambda\left(X_\sigma^{N,\lambda}\right) + h_\sigma^N\left(X_\sigma^{N,\lambda}\right)\right)dt + B_\sigma^N\left(X_\sigma^{N,\lambda}\right)dW_t, & \mathcal{O}\times(0,T), \\ X_\sigma^{N,\lambda} = 0, & \partial\mathcal{O}\times(0,T), \\ X_\sigma^{N,\lambda}(0) = b_\sigma^0, & \mathcal{O}\times\{0\}, \end{cases} \tag{10}$$

   and

   $$\begin{cases} dX^{N,\lambda} = \left(\Delta\mathcal{D}^\lambda\left(X^{N,\lambda}\right) + h^N\left(X^{N,\lambda}\right)\right)dt + B^N\left(X^{N,\lambda}\right)dW_t, & \mathcal{O}\times(0,T), \\ X^{N,\lambda} = 0, & \partial\mathcal{O}\times(0,T), \\ X^{N,\lambda}(0) = u_0, & \mathcal{O}\times\{0\}. \end{cases} \tag{11}$$

To compute the distance between the two associated solutions, we write

$$\mathbb{E}\left|X_\sigma^N(t) - X^N(t)\right|_{-1}^2 \leq C\Big(\mathbb{E}\left|X_\sigma^N(t) - X_\sigma^{N,\lambda}(t)\right|_{-1}^2 + \mathbb{E}\left|X_\sigma^{N,\lambda}(t) - X^{N,\lambda}(t)\right|_{-1}^2 + \mathbb{E}\left|X^{N,\lambda}(t) - X^N(t)\right|_{-1}^2\Big),$$

for some positive constant C and for all $t \in [0,T]$.

3. In order to pass to the limit for $\sigma \to 0$ in the relation above it's sufficient to show that

$$\mathbb{E}\left|X_\sigma^N(t) - X_\sigma^{N,\lambda}(t)\right|_{-1}^2 \to 0, \quad \text{uniformly in } \sigma, \text{ for } \lambda \to 0, \tag{12}$$

$$\mathbb{E}\left|X^{N,\lambda}(t) - X^N(t)\right|_{-1}^2 \to 0, \quad \text{for } \lambda \to 0, \tag{13}$$

and

$$\mathbb{E}\left|X_\sigma^{N,\lambda}(t) - X^{N,\lambda}(t)\right|_{-1}^2 \to 0, \quad \text{for } \sigma \to 0 \text{ and for all } \lambda \text{ fixed.} \tag{14}$$

A similar method was developed in [5] and [6] for the stochastic porous media equation. For this reason we shall avoid some details.

3. (a) In order to get the first uniform convergence we shall apply the Itô formula to the H^{-1}-norm of the difference of solutions to the equations (7) and (10), on the time interval $[0,t]$ (for $t \leq T$). By further taking the expectation, we get

$$\begin{aligned}
&\mathbb{E}\left|X_\sigma^N(t) - X_\sigma^{N,\lambda}(t)\right|_{-1}^2 + \mathbb{E}\int_0^t \int_{\mathcal{O}} \left(\mathcal{D}_\sigma\left(X_\sigma^N\right) - \mathcal{D}_\sigma^\lambda\left(X_\sigma^{N,\lambda}\right)\right)\left(X_\sigma^N - X_\sigma^{N,\lambda}\right) d\xi ds \\
= \;& \mathbb{E}\int_0^t \left\langle h_\sigma^N\left(X_\sigma^N\right) - h_\sigma^N\left(X_\sigma^{N,\lambda}\right), X_\sigma^N - X_\sigma^{N,\lambda}\right\rangle_{-1} ds + C\mathbb{E}\int_0^t \left|X_\sigma^N - X_\sigma^{N,\lambda}\right|_{-1}^2 ds.
\end{aligned}$$

To keep the expression short enough, we have not specified the dependence on space variables ξ and on the integration times s, but this is unlikely to cause any confusion. By using the properties of h_σ^N and the Gronwall inequality, we get

$$\mathbb{E}\left|X_\sigma^N(t) - X_\sigma^{N,\lambda}(t)\right|_{-1}^2 + \mathbb{E}\int_0^t \int_{\mathcal{O}} \left(\mathcal{D}_\sigma\left(X_\sigma^N\right) - \mathcal{D}_\sigma^\lambda\left(X_\sigma^{N,\lambda}\right)\right)\left(X_\sigma^N - X_\sigma^{N,\lambda}\right) d\xi ds \leq 0.$$

We have

$$\begin{aligned}
&\mathbb{E}\int_0^t \int_{\mathcal{O}} \left(\mathcal{D}_\sigma\left(X_\sigma^N\right) - \mathcal{D}_\sigma^\lambda\left(X_\sigma^{N,\lambda}\right)\right)\left(X_\sigma^N - X_\sigma^{N,\lambda}\right) d\xi ds \\
&= \mathbb{E}\int_0^t \int_{\mathcal{O}} \left(\mathcal{D}_\sigma\left(X_\sigma^N\right) - \mathcal{D}_\sigma\left((1+\lambda\mathcal{D}_\sigma)^{-1}\left(X_\sigma^{N,\lambda}\right)\right)\right)\left(X_\sigma^N - X_\sigma^{N,\lambda}\right) d\xi ds \\
&- \lambda\mathbb{E}\int_0^t \int_{\mathcal{O}} X_\sigma^{N,\lambda}\left(X_\sigma^N - X_\sigma^{N,\lambda}\right) d\xi ds \\
&\geq \mathbb{E}\int_0^t \int_{\mathcal{O}} \left(\mathcal{D}_\sigma\left(X_\sigma^N\right) - \mathcal{D}_\sigma\left((1+\lambda\mathcal{D}_\sigma)^{-1}\left(X_\sigma^{N,\lambda}\right)\right)\right) \\
&\qquad \times \left(X_\sigma^N - (1+\lambda\mathcal{D}_\sigma)^{-1}\left(X_\sigma^{N,\lambda}\right) + (1+\lambda\mathcal{D}_\sigma)^{-1}\left(X_\sigma^{N,\lambda}\right) - X_\sigma^{N,\lambda}\right) d\xi ds \\
&- C\lambda\mathbb{E}\int_0^t \int_{\mathcal{O}} \left[\left|X_\sigma^{N,\lambda}\right|^2 + \left|X_\sigma^N\right|^2\right] d\xi ds.
\end{aligned}$$

At this point, let us note that $x = \frac{\lambda}{1+\lambda^2}\mathcal{D}_\sigma^\lambda(x) + \frac{1}{1+\lambda^2}(1+\lambda\mathcal{D}_\sigma)^{-1}(x)$ which yields

$$x - (1+\lambda\mathcal{D}_\sigma)^{-1} = \frac{\lambda}{1+\lambda^2}\mathcal{D}_\sigma^\lambda(x) - \frac{\lambda^2}{1+\lambda^2}(1+\lambda\mathcal{D}_\sigma)^{-1}(x).$$

As a consequence, by invoking the monotonicity of $\mathcal{D}_\sigma$, and by using at the end the fact that we always have for the resolvent the following relation $\left|(1+\lambda\mathcal{D}_\sigma)^{-1}(X_\sigma^{N,\lambda})\right|_2^2 \leq \left|X_\sigma^{N,\lambda}\right|_2^2$, it follows that

$$\begin{aligned}
&\mathbb{E}\int_0^t\int_{\mathcal{O}}\left(\mathcal{D}_\sigma\left(X_\sigma^N\right) - \mathcal{D}_\sigma^\lambda\left(X_\sigma^{N,\lambda}\right)\right)\left(X_\sigma^N - X_\sigma^{N,\lambda}\right)d\xi ds\\
&\geq \mathbb{E}\int_0^t\int_{\mathcal{O}}\left(\mathcal{D}_\sigma\left(X_\sigma^N\right) - \mathcal{D}_\sigma\left((1+\lambda\mathcal{D}_\sigma)^{-1}\left(X_\sigma^{N,\lambda}\right)\right)\right)\left((1+\lambda\mathcal{D}_\sigma)^{-1}\left(X_\sigma^{N,\lambda}\right) - X_\sigma^{N,\lambda}\right)d\xi ds\\
&\geq -\mathbb{E}\int_0^t\int_{\mathcal{O}}\left(\left|\mathcal{D}_\sigma\left(X_\sigma^N\right)\right| + \left|\mathcal{D}_\sigma^\lambda\left(X_\sigma^{N,\lambda}\right)\right| + \lambda\left|X_\sigma^{N,\lambda}\right|\right)\\
&\qquad\times\left(\frac{\lambda}{1+\lambda^2}\left|\mathcal{D}_\sigma^\lambda\left(X_\sigma^{N,\lambda}\right)\right| + \frac{\lambda^2}{1+\lambda^2}\left|(1+\lambda\mathcal{D}_\sigma)^{-1}\left(X_\sigma^{N,\lambda}\right)\right|\right)d\xi ds\\
&\geq -\lambda C\mathbb{E}\int_0^t\int_{\mathcal{O}}\left(\left|\mathcal{D}_\sigma^\lambda\left(X_\sigma^{N,\lambda}\right)\right|^2 + \left|\mathcal{D}_\sigma\left(X_\sigma^N\right)\right|^2 + \left|X_\sigma^{N,\lambda}\right|^2\right)d\xi ds,
\end{aligned}$$

where C is a constant independent of λ.

By using the a similar argument to the one from [5] or [6] we have the following estimates.

Proposition 5 *Given a fixed time horizon $T > 0$, there exists a generic constant C depending on T, on the bounds and Lipschitz-constants of h^N and B^N and the initial datum, but independent of the approximating parameters $\sigma, \lambda > 0$ such that, for every $0 \leq t \leq T$,*

$$\mathbb{E}\int_0^t\int_{\mathcal{O}}\left[\left|\mathcal{D}_\sigma^\lambda\left(X_\sigma^{N,\lambda}\right)\right|^2 + \left|\mathcal{D}_\sigma\left(X_\sigma^N\right)\right|^2 + \left|X_\sigma^{N,\lambda}\right|^2 + \left|X_\sigma^N\right|^2\right]d\xi ds < C.$$

where C is a constant depending only on the initial condition on T.

For our readers' convenience we shall briefly sketch the proof in the Appendix, aiming especially at clarifying the contribution of the reaction-diffusion term that is not present in the papers mentioned above. For the time being, we go back to the proof our main result.

3. (b) The second convergence is based on the same arguments and can be formally seen as the case when $\sigma = 0$.

3. (c) In order to complete the proof, we only need to show that

$$\mathbb{E}\left|X_\sigma^{N,\lambda}(t) - X^{N,\lambda}(t)\right|_{-1}^2 \to 0, \quad \text{for } \sigma \to 0 \text{ and for all } \lambda \text{ fixed.} \tag{15}$$

By taking the difference between the two solutions, and applying Itô's formula to the squared norm $|\cdot|_{-1}^2$, we get that

$$\begin{aligned}
&\mathbb{E}\left|X_\sigma^{N,\lambda}(t) - X^{N,\lambda}(t)\right|_{-1}^2 + \mathbb{E}\int_0^t\int_{\mathcal{O}}\left(\mathcal{D}_\sigma^\lambda\left(X_\sigma^{N,\lambda}\right) - \mathcal{D}^\lambda\left(X^{N,\lambda}\right)\right)\left(X_\sigma^{N,\lambda} - X^{N,\lambda}\right)d\xi ds\\
&= \mathbb{E}\int_0^t\left\langle h_\sigma^N\left(X_\sigma^{N,\lambda}\right) - h^N\left(X^{N,\lambda}\right), X_\sigma^{N,\lambda} - X^{N,\lambda}\right\rangle_{-1}ds \qquad (16)\\
&+ C\mathbb{E}\int_0^t\left|B_\sigma^N\left(X_\sigma^{N,\lambda}\right) - B^N\left(X^{N,\lambda}\right)\right|_{\mathcal{L}_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))}^2 ds.
\end{aligned}$$

We shall study each term as follows. Owing to the monotonicity of $\mathcal{D}_\sigma^\lambda$, followed by Cauchy-Schwartz's inequality, we have

$$\begin{aligned}
& \mathbb{E}\int_0^t\int_{\mathcal{O}}\left(\mathcal{D}_\sigma^\lambda\left(X_\sigma^{N,\lambda}\right)-\mathcal{D}^\lambda\left(X^{N,\lambda}\right)\right)\left(X_\sigma^{N,\lambda}-X^{N,\lambda}\right)d\xi ds \\
\geq\ & \mathbb{E}\int_0^t\int_{\mathcal{O}}\left(\mathcal{D}_\sigma^\lambda\left(X^{N,\lambda}\right)-\mathcal{D}^\lambda\left(X^{N,\lambda}\right)\right)\left(X_\sigma^{N,\lambda}-X^{N,\lambda}\right)d\xi ds \\
\geq\ & -\left(\mathbb{E}\int_0^t\int_{\mathcal{O}}\left|\mathcal{D}_\sigma^\lambda\left(X^{N,\lambda}\right)-\mathcal{D}^\lambda\left(X^{N,\lambda}\right)\right|^2 d\xi ds\right)^{1/2}\left(\mathbb{E}\int_0^t\int_{\mathcal{O}}\left|X_\sigma^{N,\lambda}-X^{N,\lambda}\right|^2 d\xi ds\right)^{1/2}.
\end{aligned}$$

The reader is recalled the L^2 bounds given by Proposition 5, i.e.

$$\left(\mathbb{E}\int_0^t\int_{\mathcal{O}}\left|X_\sigma^{N,\lambda}-X^{N,\lambda}\right|^2 d\xi ds\right)^{1/2}\leq C.$$

Therefore,

$$\begin{aligned}
& \mathbb{E}\int_0^t\int_{\mathcal{O}}\left(\mathcal{D}_\sigma^\lambda\left(X_\sigma^{N,\lambda}\right)-\mathcal{D}^\lambda\left(X^{N,\lambda}\right)\right)\left(X_\sigma^{N,\lambda}-X^{N,\lambda}\right)d\xi ds \\
\geq\ & -C\left(\mathbb{E}\int_0^t\int_{\mathcal{O}}\left|\mathcal{D}_\sigma^\lambda\left(X^{N,\lambda}\right)-\mathcal{D}^\lambda\left(X^{N,\lambda}\right)\right|^2 d\xi ds\right)^{1/2}.
\end{aligned}$$

On the other hand, owing to the Lipschitz-continuity of h_σ^N (uniformly in σ) and to the continuous inclusion $L^2\subset H^{-1}$, one can write

$$\begin{aligned}
& \mathbb{E}\int_0^t\left\langle h_\sigma^N\left(X_\sigma^{N,\lambda}\right)-h^N\left(X^{N,\lambda}\right),X_\sigma^{N,\lambda}-X^{N,\lambda}\right\rangle_{-1}ds \\
\leq\ & \mathbb{E}\int_0^t\left\langle h_\sigma^N\left(X_\sigma^{N,\lambda}\right)-h_\sigma^N\left(X^{N,\lambda}\right),X_\sigma^{N,\lambda}-X^{N,\lambda}\right\rangle_{-1}ds \\
& +\mathbb{E}\int_0^t\left\langle h_\sigma^N\left(X^{N,\lambda}\right)-h^N\left(X^{N,\lambda}\right),X_\sigma^{N,\lambda}-X^{N,\lambda}\right\rangle_{-1}ds \\
\leq\ & C\mathbb{E}\int_0^t\left|X_\sigma^{N,\lambda}-X^{N,\lambda}\right|_{-1}^2 ds+\mathbb{E}\int_0^t\left|h_\sigma^N\left(X^{N,\lambda}\right)-h^N\left(X^{N,\lambda}\right)\right|_2\left|X_\sigma^{N,\lambda}-X^{N,\lambda}\right|_{-1}ds \\
\leq\ & C\left(\mathbb{E}\int_0^t\left|X_\sigma^{N,\lambda}-X^{N,\lambda}\right|_{-1}^2 ds+\mathbb{E}\int_0^t\left|h_\sigma^N\left(X^{N,\lambda}\right)-h^N\left(X^{N,\lambda}\right)\right|_2^2 ds\right).
\end{aligned}$$

Again, we insist on the generic character of C changing from one line to another (but still kept independent of the varying parameters σ,λ). With a similar argument applied to the last term of (16), one gets

$$\begin{aligned}
& C\mathbb{E}\int_0^t\left|B_\sigma^N\left(X_\sigma^{N,\lambda}\right)-B^N\left(X^{N,\lambda}\right)\right|_{\mathcal{L}_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))}^2 ds \\
\leq\ & C\left(\mathbb{E}\int_0^t\left|X_\sigma^{N,\lambda}-X^{N,\lambda}\right|_{-1}^2 ds+\mathbb{E}\int_0^t\left|B_\sigma^N\left(X^{N,\lambda}\right)-B^N\left(X^{N,\lambda}\right)\right|_2^2 ds\right).
\end{aligned}$$

By going back to (16) and replacing the previous calculus we obtain that

$$\begin{aligned}
& \mathbb{E}\left|X_\sigma^{N,\lambda}(t)-X^{N,\lambda}(t)\right|_{-1}^2 \\
& \leq C\mathbb{E}\int_0^t\left|X_\sigma^{N,\lambda}-X^{N,\lambda}\right|_{-1}^2 ds+C\Big[\left(\mathbb{E}\int_0^t\int_{\mathcal{O}}\left|\mathcal{D}_\sigma^\lambda\left(X_\sigma^{N,\lambda}\right)-\mathcal{D}^\lambda\left(X_\sigma^{N,\lambda}\right)\right|^2 d\xi ds\right)^{1/2} \\
& +\mathbb{E}\int_0^t\left|h_\sigma^N\left(X^{N,\lambda}\right)-h^N\left(X^{N,\lambda}\right)\right|_2^2 ds+\mathbb{E}\int_0^t\left|B_\sigma^N\left(X^{N,\lambda}\right)-B^N\left(X^{N,\lambda}\right)\right|_2^2 ds.\Big]
\end{aligned}\tag{17}$$

In order to complete proof, we shall use Gronwall's inequality and then pass to the limit for $\sigma \to 0$ with each λ and N fixed, in the later three terms from the right hand side of the previous relation.

- To deal with the term involving $\mathcal{D}$, we use the convergence of the resolvents of $\mathcal{D}_\sigma$ to the resolvent of $\mathcal{D}$. We have that

$$\left|\mathcal{D}_\sigma^\lambda\left(X^{N,\lambda}\right)-\mathcal{D}^\lambda\left(X^{N,\lambda}\right)\right|^2 \to 0, \text{ a.e. on } \Omega \times (0,T) \times \mathcal{O},$$

for $\sigma \to 0$ with each λ and N fixed.

Since

$$\left|\mathcal{D}_\sigma^\lambda\left(X^{N,\lambda}\right)-\mathcal{D}^\lambda\left(X^{N,\lambda}\right)\right|^2 \leq C(\lambda, N)\left|X^{N,\lambda}\right|^2$$

and

$$C(\lambda, N)\,\mathbb{E}\int_0^t\int_{\mathcal{O}}\left|X^{N,\lambda}\right|^2 d\xi ds < C$$

we get from the Lebesgue dominated convergence theorem that

$$\left(\mathbb{E}\int_0^t\int_{\mathcal{O}}\left|\mathcal{D}_\sigma^\lambda\left(X_\sigma^{N,\lambda}\right)-\mathcal{D}^\lambda\left(X_\sigma^{N,\lambda}\right)\right|^2 d\xi ds\right)^{1/2} \to 0, \text{ for } \sigma \to 0.$$

- Keeping in mint the form of the operators h_σ^N and B_σ^N, we can easly show by the same argument based on the Lebesgue dominated convergence theorem that

$$\mathbb{E}\int_0^t\left|h_\sigma^N\left(X^{N,\lambda}\right)-h^N\left(X^{N,\lambda}\right)\right|_2^2 ds \to 0, \text{ for } \sigma \to 0$$

and

$$\mathbb{E}\int_0^t\left|B_\sigma^N\left(X^{N,\lambda}\right)-B^N\left(X^{N,\lambda}\right)\right|_2^2 ds \to 0, \text{ for } \sigma \to 0.$$

This implies the convergence (14) and the proof of the theorem is complete.

■

In order to obtain the result concerning the equation (1) it is sufficient to pass to the limit for $N \to \infty$ in the equation

$$\begin{cases} dX^N = \left(\Delta\mathcal{D}\left(X^N\right)+h^N\left(X^N\right)\right)dt + B^N\left(X^N\right)dW_t, & \mathcal{O}\times(0,T), \\ X^N = 0, & \partial\mathcal{O}\times(0,T), \\ X^N(0) = u_0, & \mathcal{O}\times\{0\}. \end{cases}$$

To this purpose we consider the limit equation

$$\begin{cases} dX = \left(\Delta\mathcal{D}(X)+h(X)\right)dt + B(X)\,dW_t, & \mathcal{O}\times(0,T), \\ X = 0, & \partial\mathcal{O}\times(0,T), \\ X(0) = u_0, & \mathcal{O}\times\{0\}. \end{cases}$$

Now we can easily prove the following result.

Theorem 6 *Under the assumptions above we have that $X^N \to X$ for $\sigma \to 0$ in $C\left([0,T];L^2\left(\Omega;H^{-1}(\mathcal{O})\right)\right)$.*

Proof. The result is obtained directly from the construction of h^N and B^N, by tacking the difference between the two equations above and applying the Itô formula in $H^{-1}(\mathcal{O})$. ■

4 Appendix

4.1 Proof of Proposition 5

Proof of Proposition 5. One shall first need to take another approximation of the following type

$$(18) \quad \begin{cases} dX_\sigma^{N,\lambda,\varepsilon} = \left(-A_\sigma^{\lambda,\varepsilon}\left(X_\sigma^{N,\lambda,\varepsilon}\right) + h_\sigma^N\left(X_\sigma^{N,\lambda,\varepsilon}\right)\right)dt + B_\sigma^N\left(X_\sigma^{N,\lambda,\varepsilon}\right)dW_t, & \mathcal{O}\times(0,T), \\ X_\sigma^{N,\lambda,\varepsilon} = 0, & \partial\mathcal{O}\times(0,T), \\ X_\sigma^{N,\lambda,\varepsilon}(0) = b_\sigma^0, & \mathcal{O}\times\{0\}, \end{cases}$$

where $A_\sigma^{\lambda,\varepsilon}$ is the Yosida approximation of the operator $A_\sigma^\lambda = -\Delta\mathcal{D}_\sigma^\lambda$ in the Hilbert space $H^{-1}(\mathcal{O})$ which is defined by

$$A_\sigma^{\lambda,\varepsilon}(u) = \frac{1}{\varepsilon}\left(u - J_\varepsilon(u)\right) = A_\sigma^\lambda\left(\left(1+\varepsilon A_\sigma^\lambda\right)^{-1}(u)\right),$$

for $\varepsilon > 0,\ u \in H^{-1}(\mathcal{O})$, and where $J_\varepsilon(u) = \left(1+\varepsilon A_\sigma^\lambda\right)^{-1}(u)$.

Keeping in mind the properties of $\mathcal{D}_\sigma^\lambda$, the reader is invited to notice that the operator $A_\sigma^{\lambda,\varepsilon}$ is Lipschitz in $H^{-1}(\mathcal{O})$ and also in $L^2(\mathcal{O})$. For this reason one can use classical existence theory for the equation above in both spaces where the drift is Lipschitz.

By applying the Itô formula with the $L^2(\mathcal{O})$ norm, we get by using the same argument as in Lemma 3.1 from [3] that

$$\mathbb{E}\left|X_\sigma^{N,\lambda,\varepsilon}(t)\right|_2^2 \le C\left|u_0\right|_2^2$$

where C is a constant independent of N, λ, ε and σ. Therefore, we have as in Lemma 3.1 from [3] that

$$\underset{t\in[0,T]}{ess\sup}\mathbb{E}\left|X_\sigma^{N,\lambda}(t)\right|_2^2 \le C\left|u_0\right|_2^2.$$

Finally, from the properties of the Yosida approximation, and keeping in mind that the operator $\mathcal{D}_\sigma$ is Lipschitz, we have that

$$\begin{aligned} \mathbb{E}\int_0^t\int_{\mathcal{O}}\left|\mathcal{D}_\sigma^\lambda\left(X_\sigma^{N,\lambda}\right)\right|^2 d\xi ds &\le \mathbb{E}\int_0^t\int_{\mathcal{O}}\left|\mathcal{D}_\sigma\left(X_\sigma^{N,\lambda}\right)\right|^2 d\xi ds \\ &\le C\mathbb{E}\int_0^t\int_{\mathcal{O}}\left|X_\sigma^{N,\lambda}\right|^2 d\xi ds \le C. \end{aligned}$$

The properties of the operator $\mathcal{D}_\sigma$ are used to get

$$\mathbb{E}\int_0^t\int_{\mathcal{O}}\left|\mathcal{D}_\sigma\left(X_\sigma^N\right)\right|^2 d\xi ds < C$$

with C also a constant independent of λ. The proof of the proposition is now complete. ■

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