Optimality conditions for stochastic porous media equation under general growth conditions

Ioana Ciotir, Nicolas Forcadel, Hasnaa Zidani
LMI, Normandy University, INSA de Rouen Normandy,
76000 Rouen, France

March 31, 2023

Abstract

We study an optimal control problem for a stochastic porous media equation. More precisely we prove existence of an optimal control and to obtain first order necessary conditions (Pontryagin principle) for this type of equations.

Keywords: stochastic porous media, Wiener process, Pontryagin principle, optimal control
2020 MSC: 60H15, 76S05, 35Q93

1 Introduction

In this work we are interested to solve an optimal control problem for a stochastic porous media equation. The main goal is to prove existence of an optimal control and to obtain first order necessary conditions (Pontryagin principle) for this type of equations.

To this purpose we shall start by introducing the following stochastic differential equation

$$\begin{cases} dX - \Delta\Psi(X)\,dt + I_{\mathcal{O}_0}(\xi)\,udt = \sigma(X)\,dW(t), & (0,T)\times\mathcal{O}\times\Omega \\ \Psi(X) = 0, & (0,T)\times\partial\mathcal{O}\times\Omega \\ X(0) = x \in H^{-1}(\mathcal{O}), & \mathcal{O}\times\Omega \end{cases} \tag{1}$$

where $\mathcal{O}$ is an open, bounded domain in $\mathbb{R}^d$, $d \geq 1$, with smooth boundary $\partial\mathcal{O}$ and $\mathcal{O}_0$ is a convex subset of $\mathcal{O}$ and $(\Omega, \mathcal{F}, , \mathbb{P})$ a probability space.

The solution $X(t,\xi,\omega)$ to (1) is a function of $t \in [0,\infty)$, $\xi \in \mathcal{O}$ and $\omega \in \Omega$ but in the following it will be simply written $X(t)$ omitting ξ and ω.

Notations

We shall use the following classical notations. We denote by $H_0^1(\mathcal{O})$ the classical Sobolev space endowed with the usual norm $|\cdot|_1$ and by $H^{-1}(\mathcal{O})$ its dual endowed with the norm

$$|x|_{-1} = \left|(-\Delta)^{-1} x\right|_1, \text{ for } \forall x \in H^{-1}(\mathcal{O}).$$

Note that $(-\Delta)^{-1} x = y$ is the solution of the homogeneous Dirichlet problem $x = -\Delta y$ in $\mathcal{O}$ and the scalar product in $H^{-1}(\mathcal{O})$ is

$$\langle x, z\rangle_{-1} = \left\langle (-\Delta)^{-1} x, z\right\rangle_2, \text{ for } \forall x,\ z \in H_0^1(\mathcal{O}).$$

For two Hilbert spaces H_1 and H_2, we denote by $L_2(H_1, H_2)$ the space of Hilbert-Schmidt operators from H_1 to H_2.

Hypotheses

The equation above is defined as follows:

(H_1) We consider the cylindrical Wiener process on $L^2(\mathcal{O})$ as the series

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) e_k, \quad t \geq 0,$$

where $\{\beta_k\}_k$ is a sequence of mutually independent Brownian motions on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ and $\{e_k\}_k$ is an orthonormal basis in $L^2(\mathcal{O})$ of eigenfunctions of the Laplace operator with homogeneous Dirichlet conditions. We denote by $\{\lambda_k\}_k$ the corresponding sequence of eigenvalues

$$\Delta e_k = -\lambda_k e_k, \quad k \in \mathbb{N}.$$

(H_2) The operator σ is linear and defined as

$$\langle \sigma(x), h\rangle_2 = \sum_{k=1}^{\infty} \mu_k \langle h, e_k\rangle_2 x e_k, \quad \forall x \in H^{-1}(\mathcal{O}),\ h \in L^2(\mathcal{O}),$$

where $\{\mu_k\}_k$ is a sequence of positive numbers.

Keeping in mind the form of the Wiener process W, one can easily see that

$$\sigma(X) dW(t) = \sum_{k=1}^{\infty} \mu_k X e_k d\beta_k(t).$$

Furthermore, throughout this paper we shall assume that

$$\sum_{k=1}^{\infty} \mu_k^2 \lambda_k^2 = C < \infty.$$

This implies that

$$\sum_{k=1}^{\infty} \mu_k^2 \left|x e_k\right|_{-1}^2 \leq C_1 \sum_{k=1}^{\infty} \mu_k^2 \lambda_k^2 \left|x\right|_{-1}^2 \leq C_2 \left|x\right|_{-1}^2$$

for $\forall x \in H^{-1}(\mathcal{O})$, and therefore $\sigma(x) \in L_2\left(L^2(\mathcal{O}); H^{-1}(\mathcal{O})\right)$ so σ is well defined.

(H_3) The function $I_{\mathcal{O}_0} : \mathcal{O} \to \mathbb{R}$ is the indicator of the set $\mathcal{O}_0$.

(H_4) $\Psi \in C^1(\mathbb{R})$ is a maximal monotone operator which satisfies the following conditions

- $\Psi'(r) \leq \alpha_1 |r|^{m-1} + \alpha_2, \quad \forall r \in \mathbb{R}$,
- $\Psi(0) = 0$,
- $(\Psi(r) - \Psi(s))(r - s) \geq \alpha_3 |r - s|^{m+1}$, $\forall r \in \mathbb{R}$,
  for some constants $\alpha_i > 0$, $i = 1, 2, 3$ and for $m \geq 1$ fixed.

Remark 1 *A standard example is $\Psi(r) = a|r|^{m-1} r$ where $m \in \mathbb{R}$ and $a > 0$, $b \geq 0$. Note that the last assumption which is a generalized strong monotonicity condition, is equivalent to the fact that the inverse of the function Ψ is Hölder continuous with exponent $\frac{1}{m}$.*

The stochastic porous media equation was intensively studied recently in different frameworks (see [2] for a monograph on the subject). According to the order of growth of the operator Ψ, the porous media equation describe different phenomena going from slow diffusion, for an over-unit orders, to fast diffusion, for a sub-unit orders, and even super-fast diffusion for negative orders. In our case the equations concerns the case of a slow diffusion which describes, for example, the infiltration of a water in a porous media like the soil. For this reason, this type of equation can be used, for example the in the modelization of the erosion of a calcar cliff.

Concerning the optimal control problems in different other contexts, one can see [5], [9], [10]. For deterministic porous media equation we refer to [7]. Concerning the stochastic case one can see [8] for an optimal control problem for a general stochastic partial differential equation which doesn't cover the case of a porous media equation. A state-constrained porous media control systems with application to stabilization was studied in [6].

To the best of our knowledge, the present work is the first which gives necessary optimality conditions for stochastic porous media equations.

The optimal control problem

We can write equation (1) equivalently as

$$\begin{cases} dX + A(X)\,dt + B(u)\,dt = \sigma(X)\,dW(t), & t \in (0,T), \\ X(0) = x, \end{cases} \tag{2}$$

where

$$\begin{aligned} A &: H^{-1}(\mathcal{O}) \to H^{-1}(\mathcal{O}) \\ A(r) &= -\Delta\Psi(r), \quad \forall r \in D(A) \end{aligned}$$

for

$$D(A) = \left\{ r \in H^{-1}(\mathcal{O}) \cap L^1(\mathcal{O}) \,\middle|\, \Psi(r) \in H_0^1(\mathcal{O}) \right\}.$$

We define

$$\begin{aligned} B &: \mathcal{U} \longrightarrow L^p(\mathcal{O}) \\ B(u) &= u\, I_{\mathcal{O}_0} \in L^p(\mathcal{O}), \quad \text{for } p \geq \max\{2m+1, 4\}. \end{aligned}$$

Note that $B(u) \in H^{-1}(\mathcal{O})$ and

$$\begin{aligned} B(u) &: H_0^1(\mathcal{O}) \to \mathbb{R} \\ B(u)\varphi &= \int_{\mathcal{O}_0} u(\xi)\,\varphi(\xi)\,d\xi, \quad \forall \varphi \in H_0^1(\mathcal{O}). \end{aligned}$$

Here $u \in \mathcal{U}$ is the control and

$$\begin{aligned} \mathcal{U} = \; & \{ u \in (0,T) \times \Omega \times \mathcal{O} \longrightarrow U \mid u \in L^\infty(0,T; L^p(\Omega; L^p(\mathcal{O}))) \\ & \text{and } u \in U \text{ which is closed and convex} \}. \end{aligned}$$

In order to construct the optimal control problem we should consider that the equation (2) is assumed to be subject to

$$\underset{u \in \mathcal{U}}{Min}\, \mathbb{E} \left(\int_0^T \int_{\mathcal{O}} l(X(t,\xi), u(t,\xi))\, d\xi dt + \int_{\mathcal{O}} g(X(T))\, d\xi \right).$$

We add the following assumptions for l and g.

(H_5) $l, g \in C^1(\mathbb{R})$ and

$$|l(x,u)| \leq C\left(|x|^2 + |u|^2 + 1 \right), \quad x, u \in \mathbb{R}.$$

2 Existence of an admissible pair

We can check that, by adapting the classical theory, under the assumptions mentioned above, equation (2) has a unique solution in the sense of the definition below.

Definition 2 *Equation (1) has also a unique solution in the sense of Definition 3.1.2 from [2], which means that*

$$X \in L^2\left(\Omega; C\left([0,T]; H^{-1}(\mathcal{O})\right)\right) \cap L^2\left(\Omega \times (0,T) \times \mathcal{O}\right)$$

for some $m \geq 1$ and

$$\begin{aligned} \langle X(t), e_j \rangle_{-1} &= \langle x, e_j \rangle_{-1} - \int_0^t \langle \Psi(X(s)), e_j \rangle_2 \, ds \\ &\quad + \int_0^t \langle B(u(s)), e_j \rangle_{-1} \, ds + \sum_{k=1}^{\infty} \mu_k \int_0^t \langle X(s) e_k, e_j \rangle_{-1} \, d\beta_k(s), \end{aligned} \tag{3}$$

for $\forall j \in \mathbb{N}$, $\forall t \in [0,T]$, where $\{e_k\}_k$ is on eigenbasis for $A = -\Delta$ in $H^{-1}(\mathcal{O})$.

Remark 3 *By Remark 3.1.4 from [2], we have that (3) can be equivalently written as*

$$X(t) = x - \Delta \int_0^t \Psi(X(s)) \, ds + \int_0^t B(u(s)) \, ds + \int_0^t \sigma(X(s)) \, dW(s), \quad t \in [0,T]$$

where $\Delta : H_0^1(\mathcal{O}) \longrightarrow H^{-1}(\mathcal{O})$ is taken in the sense of distributions on $\mathcal{O}$.

Indeed, one can use an argument similar with the one from Lemma 3.1 in [3] to show that $X \in L^p(\Omega \times (0,T) \times \mathcal{O})$, for $p \geq \max\{2m+1, 4\}$. Indeed, the only difference in our case comes from the term with the control.

For reader's convenience, we shall give a small lemma to explain how the control term is treated.

Lemma 4 *Under the assumption above, for each initial condition $x \in L^p(\mathcal{O})$, $p \geq \max\{2m, 4\}$, and for each control $u \in \mathcal{U}$, fixed, there is a unique solution $X \in L^\infty(0, T : L^p(\Omega; L^p(\mathcal{O})))$ to (1).*

Proof. Because of the similarities with the results from [3], we shall only sketch the proof of this lemma.

We first take the following two approximations for the equation (1). First, the operator Ψ will be approximated by

$$x \longmapsto \Psi_\lambda(x) + \lambda x$$

where Ψ_λ is the Yosida approximation of Ψ, which is

$$\Psi_\lambda(x) = \frac{1}{\lambda}\left(x - (Id + \lambda\Psi)^{-1}(x)\right), \quad \text{for } \forall x \in \mathbb{R}, \forall \lambda > 0,$$

where Id is the identity function.

Then, we denote by A_λ the corresponding porous media operator which is

$$A_\lambda(x) = -\Delta(\Psi_\lambda(x) + \lambda x)$$

with the domain

$$D(A_\lambda) = \left\{x \in H^{-1}(\mathcal{O}) \cap L^1(\mathcal{O});\ \Psi_\lambda(x) + \lambda x \in H_0^1(\mathcal{O})\right\},$$

and we take its Yosida approximation in $H^{-1}(\mathcal{O})$, i.e.

$$(A_\lambda)_\varepsilon = \frac{1}{\varepsilon}\left(Id - (Id + \varepsilon A_\lambda)^{-1}\right), \quad \varepsilon > 0.$$

By classical theory (see e.g. [4], Lemma 3.2) the approximating equation has a unique solution $X_\lambda^\varepsilon \in L^2\left(\Omega; C\left([0,T]; L^2(\mathcal{O})\right)\right)$.

By applying the Itô formula with $\varphi(x) = |x|_p^p$ we get

$$\begin{aligned}
&\mathbb{E}\varphi(X_\lambda^\varepsilon(t,\xi)) + \mathbb{E}\int_0^t\int_{\mathcal{O}}(A_\lambda)_\varepsilon(X_\lambda^\varepsilon(s,\xi))\,|X_\lambda^\varepsilon(s,\xi)|^{p-2}X_\lambda^\varepsilon(s,\xi)\,d\xi ds \\
&+\mathbb{E}\int_0^t\int_{\mathcal{O}}u(\xi)\,I_{\mathcal{O}_0}(\xi)\,|X_\lambda^\varepsilon(s,\xi)|^{p-2}X(s,\xi)\,d\xi ds \\
\leq\ & \varphi(x) + \frac{p-1}{2}C\mathbb{E}\int_0^t\int_{\mathcal{O}}|X_\lambda^\varepsilon(s,\xi)|^p\,d\xi ds.
\end{aligned}$$

Since

$$\begin{aligned}
&\mathbb{E}\int_0^t\int_{\mathcal{O}}u(\xi)\,I_{\mathcal{O}_0}(\xi)\,|X_\lambda^\varepsilon(\xi)|^{p-2}X(\xi)\,d\xi ds \\
\leq\ & \frac{1}{p}\mathbb{E}\int_0^t\int_{\mathcal{O}}|u(\xi)\,I_{\mathcal{O}_0}(\xi)|^p\,d\xi ds + \frac{p-1}{p}\mathbb{E}\int_0^t\int_{\mathcal{O}}|X_\lambda^\varepsilon(\xi)|^p\,d\xi ds,
\end{aligned}$$

one can easily get the existence result exactly as in Lemma 3.1 from [3]. ■

3 The optimality necessary conditions

In order to study the optimal control problem we shall prove the following lemmas.

Lemma 5 *Let*

$$\Gamma : L^\infty\left(0,T;L^2(\Omega;L^p(\mathcal{O}))\right) \longrightarrow L^\infty\left(0,T;L^2\left(\Omega;H^{-1}(\mathcal{O})\right)\right)$$

defined by

$$u \longmapsto \Gamma(u) = X^u.$$

This application is continuous.

Proof. We need to compute

$$\|\Gamma(u+v)-\Gamma(u)\|_{L^\infty(0,T;L^2(\Omega;H^{-1}(\mathcal{O})))}.$$

To this purpose, first we take the difference

$$\left\langle X^{u+v}(t)-X^u(t),e_j\right\rangle_{-1}+\int_0^t\int_{\mathcal{O}}\left(\Psi\left(X^{u+v}(s)\right)-\Psi\left(X^u(s)\right)\right)e_j d\xi ds$$

$$+\int_0^t\langle I_{\mathcal{O}_0}v,e_j\rangle_{-1}ds=\sum_{k=1}^{\infty}\mu_k\int_0^t\left\langle\left(X^{u+v}(s)-X^u(s)\right)e_k,e_j\right\rangle_{-1}d\beta_k(s)$$

and by using the Itô formula and after taking the expectation we get that

$$\mathbb{E}\left\|X^{u+v}(t)-X^u(t)\right\|_{-1}^2$$

$$+\mathbb{E}\int_0^t\int_{\mathcal{O}}\left(\Psi\left(X^{u+v}(s)\right)-\Psi\left(X^u(s)\right)\right)\left(X^{u+v}(s)-X^u(s)\right)d\xi ds$$

$$+\mathbb{E}\int_0^t\left\langle I_{\mathcal{O}_0}v,X^{u+v}(s)-X^u(s)\right\rangle_{-1}ds\le C\mathbb{E}\int_0^t\left\|X^{u+v}(s)-X^u(s)\right\|_{-1}^2ds.$$

By Gronwall's inequality we get that

$$\mathbb{E}\left\|X^{u+v}(t)-X^u(t)\right\|_{-1}^2\le\mathbb{E}\int_0^t\|I_{\mathcal{O}_0}v\|_{-1}^2ds\le\mathbb{E}\int_0^t\|v\|_2^2ds.$$

Indeed, since $v\in L^2(\mathcal{O})$ it is obvious that $I_{\mathcal{O}_0}v\in L^2(\mathcal{O})$ and

$$\|I_{\mathcal{O}_0}v\|_{-1}^2\le\|I_{\mathcal{O}_0}v\|_2^2\le\|v\|_2^2.$$

Therefore

$$\mathbb{E}\left\|X^{u+v}(t)-X^u(t)\right\|_{-1}^2\le\|v\|_{L^\infty(0,T;L^2(\Omega;H^{-1}(\mathcal{O})))}.$$

■

Lemma 6 *The application* Γ *is Fréchet differentiable.*

Proof. We want to prove that

$$\|\Gamma(u+\varepsilon v)-\Gamma(u)-\varepsilon\Gamma'(u)v\|_{L^\infty(0,T;L^2(\Omega;H^{-1}(\mathcal{O})))}\overset{\varepsilon\to 0}{\longrightarrow}0.$$

To this purpose we introduced the variation of $X^u=\Gamma(u)$ denoted by

$$Z^{u,v}=\lim_{\varepsilon\to 0}\frac{X^{u+\varepsilon v}-X^u}{\varepsilon}.$$

We compute

$$\left\langle\frac{X^{u+\varepsilon v}-X^u}{\varepsilon},e_j\right\rangle_{-1}+\int_0^t\int_{\mathcal{O}}\frac{\Psi\left(X^{u+\varepsilon v}\right)-\Psi\left(X^u\right)}{\varepsilon}e_j d\xi ds+\int_0^t\langle I_{\mathcal{O}_0}v,e_j\rangle_{-1}ds$$

$$= \frac{1}{\varepsilon} \sum_{k=1}^{\infty} \mu_k \int_0^t \left\langle \left(X^{u+v}(s) - X^u(s) \right) e_k, e_j \right\rangle_{-1} d\beta_k(s).$$

We have

$$\begin{aligned} I &= \int_0^t \int_{\mathcal{O}} \frac{\Psi\left(X^{u+\varepsilon v}\right) - \Psi\left(X^u\right)}{\varepsilon} e_j d\xi ds \\ &= \frac{1}{\varepsilon} \int_0^t \int_{\mathcal{O}} \int_0^1 \frac{d}{dr} \Psi\left(X^{u+\varepsilon v r}\right) e_j d\xi ds \\ &= \int_0^t \int_{\mathcal{O}} \int_0^1 \Psi'\left(X^{u+\varepsilon v r}\right) Z^{u,v} e_j dr d\xi ds \\ & \underset{\varepsilon \to 0}{\longrightarrow} \int_0^t \int_{\mathcal{O}} \Psi'\left(X^u\right) Z^{u,v} e_j d\xi ds, \end{aligned}$$

We need to compute

$$\frac{1}{\varepsilon} \sum_{k=1}^{\infty} \mu_k \int_0^t \left\langle \left(X^{u+v}(s) - X^u(s) \right) e_k, e_j \right\rangle_{-1} d\beta_k(s) = Z dW(t)$$

We get that Z is a solution to the equation (this is formal for the moment because we don't have

$$\begin{cases} dZ - \Delta\left(\Psi'(X) Z\right) dt + I_{\mathcal{O}_0} v dt = Z dW(t) \\ Z(0) = 0 \end{cases}$$

for $\forall X,\ v \in L^{\infty}\left(0,T; L^2\left(\Omega; H^{-1}(\mathcal{O})\right)\right) \cap L^2\left((0,T) \times \Omega \times \mathcal{O}\right)$ fixed.

Since the equation is linear, we have a unique solution

$$Z \in L^{\infty}\left(0,T; L^2\left(\Omega; H^{-1}(\mathcal{O})\right)\right) \cap L^2\left((0,T) \times \Omega \times \mathcal{O}\right)$$

of the form

$$\begin{aligned} & \langle Z(t), e_j \rangle_{-1} + \int_0^t \langle \Psi'\left(X^u(s)\right) Z, e_j \rangle_2 ds + \int_0^t \langle I_{\mathcal{O}_0} v, e_j \rangle_{-1} ds \qquad (4) \\ = & \sum_{k=1}^{\infty} \mu_k \int_0^t \langle Z e_k, e_j \rangle_{-1} d\beta_k(s). \end{aligned}$$

We compute now

$$\begin{aligned} & \left\langle X^{u+\varepsilon v}(t) - X^u(t) - \varepsilon Z(t), e_j \right\rangle_{-1} \\ & + \int_0^t \int_{\mathcal{O}} \left\{ \int_0^1 \Psi'\left(\theta X^{u+\varepsilon v} + (1-\theta) X^u\right) d\theta \left(X^{u+\varepsilon v} - X^u\right) e_j - \varepsilon \Psi'\left(X^u\right) Z e_j \right\} d\xi ds \\ = & \sum_{k=1}^{\infty} \mu_k \int_0^t \left\langle \left(X^{u+\varepsilon v} - X^u - \varepsilon Z \right) e_k, e_j \right\rangle_{-1} d\beta_k. \end{aligned}$$

We replace the second term from the left-hand side of the previous equality by

$$\int_0^t \int_{\mathcal{O}} \Psi'(X^u)\left(X^{u+\varepsilon v} - X^u - \varepsilon Z\right) e_j d\xi ds$$
$$+\int_0^t \int_{\mathcal{O}} \left\{\int_0^1 \Psi'\left(\theta X^{u+\varepsilon v} + (1-\theta) X^u\right) d\theta - \Psi'(X^u)\right\}\left(X^{u+\varepsilon v} - X^u\right) e_j d\xi ds.$$

We denote

$$X^{u+\varepsilon v}(t) - X^u(t) - \varepsilon Z(t) = \eta^\varepsilon(t)$$

and we have

$$\begin{aligned} & \langle \eta^\varepsilon(t), e_j\rangle_{-1} + \int_0^t \int_{\mathcal{O}} \Psi'(X^u)\, \eta^\varepsilon(s)\, e_j d\xi ds \\ & + \int_0^t \int_{\mathcal{O}} \left\{\int_0^1 \Psi'\left(\theta X^{u+\varepsilon v} + (1-\theta) X^u\right) d\theta - \Psi'(X^u)\right\}\left(X^{u+\varepsilon v} - X^u\right) e_j d\xi ds \\ = & \sum_{k=1}^{\infty} \mu_k \int_0^t \langle \eta^\varepsilon(t)\, e_k, e_j\rangle_{-1} d\beta_k(s). \end{aligned}$$

By the Itô formula we get, after taking the expectation, that

$$\begin{aligned} & \mathbb{E}\left\|\eta^\varepsilon(t)\right\|_{-1}^2 + \mathbb{E}\int_0^t \int_{\mathcal{O}} \Psi'(X^u)\left(\eta^\varepsilon(s)\right)^2 d\xi ds \\ & + \mathbb{E}\int_0^t \int_{\mathcal{O}} \left\{\int_0^1 \Psi'\left(\theta X^{u+\varepsilon v} + (1-\theta) X^u\right) d\theta - \Psi'(X^u)\right\}\left(X^{u+\varepsilon v} - X^u\right) \eta^\varepsilon(s)\, d\xi ds \\ = & \mathbb{E}\int_0^t \left\|\eta^\varepsilon(s)\right\|_{-1}^2 ds. \end{aligned}$$

Since Ψ is strongly monotone we have that $\Psi' \geq c > 0$ and by using the Gronwall inequality we have that

$$\begin{aligned} & \sup_{r\in[0,t]} \mathbb{E}\left\|\eta^\varepsilon(r)\right\|_{-1}^2 + \mathbb{E}C\int_0^t \int_{\mathcal{O}} \left|\eta^\varepsilon(s)\right|^2 d\xi ds \qquad (5) \\ \leq & \mathbb{E}\int_0^t \int_{\mathcal{O}} \left|\left\{\int_0^1 \Psi'\left(\theta X^{u+\varepsilon v} + (1-\theta) X^u\right) d\theta - \Psi'(X^u)\right\}\left(X^{u+\varepsilon v} - X^u\right) \eta^\varepsilon(s)\right| d\xi ds. \end{aligned}$$

We shall first denote

$$\widetilde{J}_\varepsilon = \int_0^1 \left(\Psi'\left(\theta X^{u+\varepsilon v} + (1-\theta) X^u\right) - \Psi'(X^u)\right) d\theta$$

and replacing in (5), we get that

$$\begin{aligned}
& \sup_{r\in[0,t]} \mathbb{E}\left\|\eta^{\varepsilon}(r)\right\|_{-1}^{2}+\mathbb{E}C\int_0^t\int_{\mathcal{O}}\left|\eta^{\varepsilon}(s)\right|^2 d\xi ds \\
\leq\ & \mathbb{E}\int_0^t\int_{\mathcal{O}}\left|\widetilde{J}_{\varepsilon}\left(X^{u+\varepsilon v}-X^{u}\right)\eta^{\varepsilon}(s)\right| d\xi ds \\
\leq\ & \frac{1}{2C}\mathbb{E}\int_0^t\int_{\mathcal{O}}\left|\widetilde{J}_{\varepsilon}\left(X^{u+\varepsilon v}-X^{u}\right)\right|^2 d\xi ds \\
& +\frac{C}{2}\mathbb{E}\int_0^t\int_{\mathcal{O}}\left|\eta^{\varepsilon}(s)\right|^2 d\xi ds,
\end{aligned}$$

and therefore

$$\sup_{r\in[0,t]} \mathbb{E}\left\|\eta^{\varepsilon}(r)\right\|_{-1}^{2}\leq\frac{1}{2C}\mathbb{E}\int_0^t\int_{\mathcal{O}}\left|\widetilde{J}_{\varepsilon}\left(X^{u+\varepsilon v}-X^{u}\right)\right|^2 d\xi ds.$$

We can easily see that

$$\begin{aligned}
& \mathbb{E}\int_0^t\int_{\mathcal{O}}\left|\widetilde{J}_{\varepsilon}\left(X^{u+\varepsilon v}-X^{u}\right)\right|^2 d\xi ds \\
\leq\ & \left(\mathbb{E}\int_0^t\int_{\mathcal{O}}\left|\widetilde{J}_{\varepsilon}\right|^q d\xi ds\right)^{1/q}\left(\mathbb{E}\int_0^t\int_{\mathcal{O}}\left|X^{u+\varepsilon v}-X^{u}\right|^r d\xi ds\right)^{1/r}
\end{aligned}$$

for $\frac{1}{2}=\frac{1}{q}+\frac{1}{r}$.

In order to use the generalized strong monotonicity assumption on Ψ we need to take $r=m+1$, we get that $q=\frac{2m+1}{m-1}$.

We calculate the first factor from the relation above

$$\begin{aligned}
\left|\widetilde{J}_{\varepsilon}\right| &= \left|\int_0^1\left(\Psi'\left(\theta X^{u+\varepsilon v}+(1-\theta)X^{u}\right)-\Psi'\left(X^{u}\right)\right)d\theta\right| \\
&\leq \int_0^1\left|\Psi'\left(\theta X^{u+\varepsilon v}+(1-\theta)X^{u}\right)-\Psi'\left(X^{u}\right)\right|d\theta \\
&\leq \int_0^1\left(\left|\theta X^{u+\varepsilon v}+(1-\theta)X^{u}\right|^{m-1}+\left|X^{u}\right|^{m-1}\right)d\theta.
\end{aligned}$$

Keeping in mind that the function $f(x)=|x|^p$ for $p\geq 1$ is convex and in our case we can assumed that $m\geq 2$, we obtain that

$$\left|\widetilde{J}_{\varepsilon}\right|\leq C\left(\left|X^{u+\varepsilon v}\right|^{m-1}+\left|X^{u}\right|^{m-1}\right),$$

for some positive constant C.

Consequently

$$\mathbb{E}\int_0^t\int_{\mathcal{O}}\left|\widetilde{J}_{\varepsilon}\right|^{\frac{2m+1}{m-1}} d\xi ds\leq C\mathbb{E}\int_0^t\int_{\mathcal{O}}\left(\left|X^{u+\varepsilon v}\right|^{2m+1}+\left|X^{u}\right|^{2m+1}\right)d\xi ds\leq C$$

since $p \geq 2m+1$. Therefore

$$\sup_{r\in[0,t]} \mathbb{E} \|\eta^{\varepsilon}(r)\|_{-1}^2 \leq C \left(\mathbb{E} \int_0^t \int_{\mathcal{O}} \left| X^{u+\varepsilon v} - X^u \right|^{m+1} d\xi ds \right)^{1/m+1}.$$

On the other hand we know that

$$\mathbb{E} \left\| X^{u+\varepsilon v} - X^u \right\|_{-1}^2 + C\mathbb{E} \int_0^t \int_{\mathcal{O}} \left| X^{u+\varepsilon v} - X^u \right|^{m+1} d\xi ds \leq \varepsilon C |v|_2^2 \underset{\varepsilon\to 0}{\longrightarrow} 0 . \quad (6)$$

Indeed, since

$$\begin{aligned} & \left\langle X^{u+\varepsilon v} - X^u, e_j \right\rangle_{-1} + \int_0^t \int_{\mathcal{O}} \left(\Psi\left(X^{u+\varepsilon v}\right) - \Psi\left(X^u\right) \right) e_j d\xi ds \\ & + \varepsilon \mathbb{E} \int_0^t \left\langle I_{\mathcal{O}_0} v, e_j \right\rangle_{-1} d\xi ds \\ = \; & \sum_{k=1}^{\infty} \int_0^t \left\langle \left(\sigma\left(X^{u+\varepsilon v}\right) - \sigma\left(X^u\right) \right) e_k, e_j \right\rangle_{-1} d\beta_k(s) \end{aligned}$$

we get by Itô's formula and the expectation that

$$\begin{aligned} & \mathbb{E} \left\| X^{u+\varepsilon v} - X^u \right\|_{-1}^2 + \mathbb{E} \int_0^t \int_{\mathcal{O}} \left(\Psi\left(X^{u+\varepsilon v}\right) - \Psi\left(X^u\right) \right) \left(X^{u+\varepsilon v} - X^u \right) d\xi ds \\ & + \varepsilon \mathbb{E} \int_0^t \left\langle I_{\mathcal{O}_0} v, X^{u+\varepsilon v} - X^u \right\rangle_{-1} ds \\ \leq \; & C\mathbb{E} \int_0^t \left| \sigma\left(X^{u+\varepsilon v}\right) - \sigma\left(X^u\right) \right|_{\mathcal{L}_2(H^{-1}(\mathcal{O}))}^2 \\ \leq \; & C\mathbb{E} \int_0^t \left\| X^{u+\varepsilon v} - X^u \right\|_{-1}^2 ds. \end{aligned}$$

By using again the Gronwall inequality and the strong monotonicity of Ψ we get (6) which leads to

$$\sup_{r\in[0,t]} \mathbb{E} \|\eta^{\varepsilon}(r)\|_{-1}^2 \underset{\varepsilon\to 0}{\longrightarrow} 0, \quad t \in [0,T].$$

The proof of the lemma is now complete. ■

The necessary conditions

Let v be an admissible control. For the cost functional

$$\widetilde{\mathcal{F}}(u) = \mathcal{F}(X,u) = \mathbb{E} \left\{ \int_0^t \int_{\mathcal{O}} l\left(X^u(t), u(t)\right) d\xi dt + \int_{\mathcal{O}} g\left(X^u(T)\right) d\xi \right\}$$

we compute

$$\begin{aligned}\widetilde{\mathcal{F}}'(u)v &= \mathcal{F}'_x(X^u,u)Z+\mathcal{F}'_u(X^u,u)v\\ &= \mathbb{E}\left\{\int_0^T\int_{\mathcal{O}} l'_x(X^u(t),u(t))Zd\xi dt+\int_0^T\int_{\mathcal{O}} l'_u(X^u(t),u(t))vd\xi dt\right.\\ &\quad\left.\int_{\mathcal{O}} g'_x(X^u(T))Z^u(T)d\xi\right\}.\end{aligned}$$

We consider the following adjoint backward SPDE in the two unknown adapted process $(p(t,\xi),q(t,\xi))\in L^2(\mathcal{O})\times L_2\left(L^2(\mathcal{O});\mathbb{R}\right)$

$$\begin{cases} dp(t,x)=-\left\{\left(\dfrac{\partial H}{dy}\right)(t,x,Y,u,p,q)+L^*p\right\}+qdW_t\\ p(T)=\dfrac{\partial g}{dx}(X(T))\end{cases}$$

which is

$$\begin{cases} dp(t,x)=(-\Psi'(X)\Delta p+l_x(X,u)+\sigma'(X)q)\,dt+qdW_t\\ p(T)=\dfrac{\partial g}{dx}(X(T)).\end{cases}$$

This equation needs to have an unique solution (see Theorem 4.2 from [1]).

We use this equation in the following calculus

$$\begin{aligned}\int_{\mathcal{O}} g'_x(X^u(T))Z^u(T)d\xi &= \int_{\mathcal{O}}\underbrace{g'_x(X^u(T))}_{=P(T)}Z^u(T)d\xi-\int_{\mathcal{O}} g'_x(X^u(T))\underbrace{Z(0)}_{=0}d\xi\\ &= \int_{\mathcal{O}} p(T)Z^u(T)d\xi-\int_{\mathcal{O}} p(0)Z^u(0)d\xi\\ &= \int_0^T d\langle p(t),Z^u(t)\rangle dt.\end{aligned}$$

We calculate the Itô product above

$$\begin{aligned}d\langle p(t),Z^u(t)\rangle_2 &= p(t)dZ^u(t)+Z^u(t)dp(t)+\langle dp,dZ^u\rangle_2\\ &= p(t)\Delta(\Psi'(X)Z^u)dt-p(t)I_{\mathcal{O}_0}vdt+p(t)\sigma'(Z^u)dW(t)\\ &\quad -Z^u(t)\Psi'(X)\Delta pdt-Z^u(t)l_x(X,u)dt-Z^u(t)\sigma'(X)qdt\\ &\quad +Z^uqdW_t+q\sigma'(X)Z^u(t)dt\\ &= -p(t)I_{\mathcal{O}_0}vdt-Z^u(t)l_x(X,u)dt.\end{aligned}$$

We replace in the last term of $\widetilde{\mathcal{F}}'(u)v$ and we get

$$\begin{aligned}&\mathbb{E}\int_{\mathcal{O}} g'_x(X^u(T))Z^u(T)d\xi\\ =\ &\mathbb{E}\int_0^T d\langle p(t),Z^u(t)\rangle_2 dt\\ =\ &\mathbb{E}\int_0^T\left(-\langle p(t),I_{\mathcal{O}_0}v\rangle_2-\langle Z^u(t),l_x(X,u)\rangle_2\right)dt.\end{aligned}$$

We obtain

$$\begin{aligned}\widetilde{\mathcal{F}}'(u)\,v &= \mathbb{E}\left\{\int_0^T\int_{\mathcal{O}} l_x\left(X^u(t),u(t)\right)Z^u(t)\,d\xi dt+\int_0^T\int_{\mathcal{O}} l_u\left(X^u(t),u(t)\right)v d\xi dt\right.\\ &\qquad\left.\int_0^T\left(-\left\langle p(t),I_{\mathcal{O}_0}v\right\rangle_2-\left\langle Z^u(t),l_x\left(X^u(t),u(t)\right)\right\rangle_2\right)dt\right\}\\ &= \mathbb{E}\left\{\int_0^T\left(\int_{\mathcal{O}} l_u'\left(X^u(t),u(t)\right)v d\xi-\left\langle p(t),I_{\mathcal{O}_0}v\right\rangle_2\right)dt\right\}.\end{aligned}$$

If $\left(\overline{X},\overline{u}\right)$ is an optimal pair, then

$$\widetilde{\mathcal{F}}'(\overline{u})(v-\overline{u})\geq 0,\quad \forall v\in\mathcal{U}.$$

We can write it as

$$\mathbb{E}\left\{\int_0^T\int_{\mathcal{O}}\left(l_u'\left(X^u(t),\overline{u}(t)\right)-p(t)\,I_{\mathcal{O}_0}\right)(v-\overline{u})\,d\xi dt\right\}\geq 0,\quad \forall v\in\mathcal{U}.$$

References

[1] A.R. Al-Hussein, *Backward stochastic partial differential equations driven by infinite-dimensional martingales and applications*, Stochastics, Taylor & Francis, 81, 6, 601-626, 2009

[2] V. Barbu, G. Da Prato, M. Röckner, *Stochastic Porous Media Equations*, Springer International Publishing, 2016.

[3] V. Barbu, G.D. Prato, M. Röckner, *Stochastic Porous Media Equations and Self-Organized Criticality.* Commun. Math. Phys. 285, 901–923, 2009.

[4] V. Barbu, G. Da Prato, M. Röckner, *Existence and Uniqueness of Nonnegative Solutions to The Stochastic Porous Media Equation.* Indiana University Mathematics Journal, 57(1), 187–211, 2008.

[5] V. Barbu, Th. Precupanu, *Convexity and optimization in Banach spaces, Mathematics and its Applications* (East European Series), Vol. 10, D. Reidel Publishing Co., 1986.

[6] I. Ciotir, D. Goreac, I. Munteanu, *State-constrained porous media control systems with application to stabilization*, Journal of Evolution Equations, 23(2), 2023.

[7] G. Marinoschi, *Functional approach to nonlinear models of water flow in soils.* Vol. 21. Dordrecht: Springer, 2006.

[8] B. Øksendal, *Optimal Control of Stochastic Partial Differential Equations*, Stochastic Analysis and Applications. 23. 165-179, 10.1081/SAP-200044467, 2005.

[9] J. Raymond, H. Zidani, *Pontryagin's principle for state-constrained control problems governed by parabolic equations with unbounded controls*, SIAM J. Control Optim., 36, 1853–1879, 1998.

[10] M. Bergounioux and H. Zidani, *Pontryagin Maximum Principle for Optimal Control of Variational Inequalities*, SIAM Journal on Control and OptimizationVol. 37, Iss. 4, 1999.