Sticky diffusion processes on networks and corresponding Mean Field Games

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Outline

Motivations

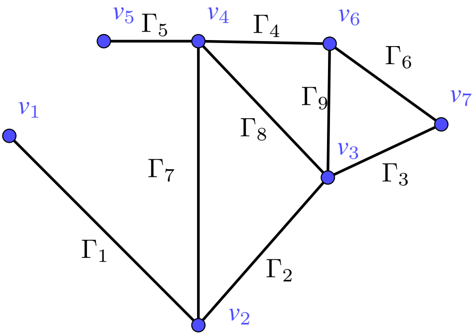

Figure: Example of network.

Markov processes

- A Markov process $(X, (\mathcal{F}_t)_{t\geq 0})$ on a locally compact Polish space E is a $\mathcal{F}_t$-adapted stochastic process X such that, when started at x,

$$\mathbb{E}_x\left[f(X(t+h)|\mathcal{F}_t\right] = \mathbb{E}_{X_t}\left[f(X(h))\right] \quad \text{for all } f \in B(E) \text{ and } x \in E.$$

- The law of the process

$$p_t(x, A) := \mathbb{P}_x\left(X(t) \in A\right) \quad \text{for all } A \in \mathcal{B}(E)$$

satisfies the Chapman-Kolmogorov identity

$$p_{t+s}(x, A) = \int_E p_t(y, A) p_s(x, dy).$$

- We obtain a positive semigroup of contractions $(P_t)_{t\geq 0}$ on $C_0(E)$ by setting

$$(P_t f)(x) = \mathbb{E}_x\left[f(X_t)\right] = \int_E f(y) p_t(x, dy).$$

If the semigroup is strongly continuous, we say that X is a Feller process.

Feller processes

- Conversely, given a strongly continuous positive semigroup of contractions $(P_t)_{t \geq 0}$ on $C_0(E)$, we can construct a Markov process.
- The Riesz-Markov theorem provides a collection of probability measures $(p_t(x, \cdot))_{t \geq 0}$ such that

$$(P_t f)(x) = \int_E f(y) p_t(x, dy).$$

- Kolmogorov's extension theorem then provides a stochastic process with law $(p_t(x, \cdot))_{t \geq 0}$ on some appropriate probability space.
- The semigroup property then ensures that the stochastic process is a Markov process.
- We recall Dynkin's formula

$$\mathbb{E}_x \left[f(X(t)) \right] - f(x) = \mathbb{E}_x \left[\int_0^t Lf(X(s)) \, ds \right]$$

for all $f \in D(L)$, where $(L, D(L))$ is the infinitesimal generator of the semigroup associated to X.

Why sticky ?

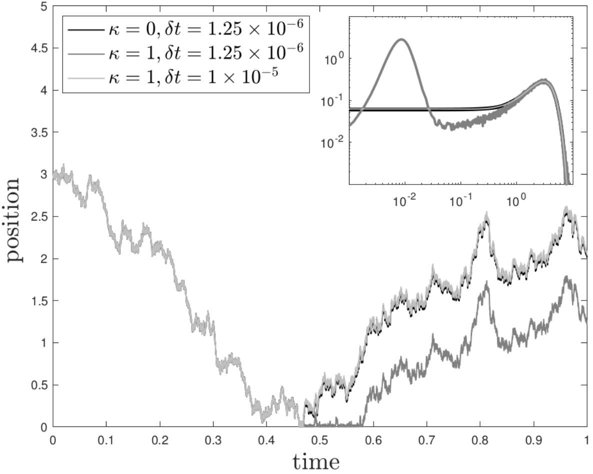

Figure: Sticky BM. From Bou-Rabee and Holmes-Cerfon (2020).

Sticky Brownian motion

- The sticky (reflected) Brownian motion is obtained by considering the Feller process whose semigroup on $C_0(\mathbb{R}_+)$ has generator $(L, D(L))$ given by

$$
\begin{aligned}
Lf(x) &= \frac{1}{2} f''(x) \quad \text{for } x > 0 \\
D(L) &= \left\{ f \in C^2((0,+\infty)) \cap C_0(\mathbb{R}_+) : \eta f''(0^+) = f'(0^+) \right\}.
\end{aligned}
$$

- The domain of the generator of reflected Brownian motion is

$$
D(L) = \left\{ f \in C^2((0,+\infty)) \cap C_0(\mathbb{R}_+) : f'(0^+) = 0 \right\}.
$$

A reflected BM B can be represented as $B(t) = |W(t)|$ for some (standard) one dimensional BM W. Tanaka's formula then implies that it satisfies

$$
dB(t) = \operatorname{sgn}(W(t)) dW(t) + d\ell_W(t) \quad t > 0,
$$

where ℓ_W is the local time of W at 0. The local time can be approximated by

$$
\ell_W(t) = \lim_{\delta \to 0} \frac{1}{\delta} \int_0^t \mathbb{1}_{(-\delta,\delta)}(W(s))\, ds \quad \text{in probability.}
$$

Sample paths of sticky Brownian motion

The sample paths of the sticky BM were constructed by Itô and McKean (1963) :

- Consider B a reflected BM and denote by ℓ_B its local time at 0.
- Define the continuous (strictly) increasing function

$$V(t) := t + \eta \ell_B(t)$$

  and consider its (right) inverse

$$V^{-1}(t) := \inf\{s > 0 : V(s) > t\}.$$

- Set $X(t) := B(V^{-1}(t))$. Then, X is a sticky BM with parameter η.

It was proved by Engelbert-Peskir (2014) and Bass (2014) that the sticky BM satisfies the stochastic differential equation

$$\begin{cases} dX(t) & = \mathbb{1}_{\{X(t)>0\}} dW(t) + \frac{1}{2} d\ell_X(t), \\ \eta \ell_X(t) & = \int_0^t \mathbb{1}_{\{X(s)=0\}} ds. \end{cases} \tag{1}$$

Moreover, the pair (X, W) is unique in law.

Properties of the sticky BM

- The property

$$\eta \ell_X(t) = \int_0^t \mathbb{1}_{\{X(s)=0\}} ds$$

implies that the sticky the set $\{t : X(t) = 0\}$ has strictly postive measure, although it has empty interior.

- Since $V(t) = t + \eta \ell_B(t) \geq t$ it follows that $V^{-1}(t) \leq t$. Therefore the sticky process X is a slowed-down version of the reflected BM.

- This slow-down can also be observed in the fact that, for small δ,

$$\mathbf{E}_{\mathsf{v}}[T_X^\delta] = \eta\delta + O(\delta^2)$$

while

$$\mathbf{E}_{\mathsf{v}}[T_B^\delta] = O(\delta^2),$$

where

$$T_X^\delta := \inf\{t \geq 0 : X(t) \geq \delta\},$$
$$T_B^\delta := \inf\{t \geq 0 : B(t) \geq \delta\}.$$

Sticky diffusion processes on $\mathbb{R}_+$

- It is possible to consider more general sticky diffusion processes on $\mathbb{R}_+$ by considering the Feller process generated by

$$Lf := \frac{1}{2}\sigma^2(x)f''(x) + b(x)f(x) \quad x > 0,$$
$$D(L) := \left\{ f \in C^2((0,+\infty)) \cap C_0(\mathbb{R}_+) : \eta Lf(0^+) = f'(0^+) \right\}.$$

- In Salins-Spiliopoulos (2017), it was established that are time-changed reflected diffusion processes ($\eta = 0$) through the Itô-McKean technique.
- They also prove that sticky diffusions solve the SDE

$$\begin{cases} dX(t) = \sigma(X(t))\mathbb{1}_{\{X(t)>0\}}dW(t) + b(X(t))\mathbb{1}_{\{X(t)>0\}}dt + \frac{1}{2}d\ell_X(t), \\ \eta\ell_X(t) = \int_0^t \mathbb{1}_{\{X(s)=0\}}ds, \end{cases}$$

for some standard BM W.

Skew sticky diffusions on $\mathbb{R}$

- Skew diffusion processes behave like a standard one-dimensional diffusion inside $\mathbb{R} \setminus \{0\}$ and, upon reaching 0, are reflected in $\mathbb{R}_+$ with probability p and in $\mathbb{R}_-$ with probability $1 - p$.
- The generator of skew sticky diffusions is given by

$$
\begin{aligned}
Lf(x) &= \frac{1}{2}\sigma^2(x)f''(x) + b(x)f'(x) \quad \text{for } x \neq 0, \\
D(L) &= \left\{ f \in C^2(\mathbb{R} \setminus \{0\}) \cap C_0(\mathbb{R}_+) : \left| \begin{array}{rl} Lf(0^+) &= Lf(0^-), \\ \eta Lf(0) &= pf'(0^+) - (1-p)f'(0^-) \end{array} \right. \right\}.
\end{aligned}
$$

- It is proved in Salins-Spiliopoulos (2017) that skew sticky diffusions are time-changed skew nonsticky diffusions ($\eta = 0$). They also satisfy the SDE

$$
\begin{cases}
dX(t) = \sigma(X(t))\mathbb{1}_{\{X(t)\neq 0\}}dW(t) + b(X(t))\mathbb{1}_{\{X(t)\neq 0\}}dt + (2p-1)d\ell_X(t), \\
\eta \ell_X(t) = \int_0^t \mathbb{1}_{\{X(s)=0\}}ds,
\end{cases}
$$

for some standard one-dimensional BM W.

Diffusion processes on an infinite junction

- We consider diffusion processes on the infinite junction

$$\Gamma = \bigcup_{i=1}^{N} \Gamma_i, \quad \Gamma_i = \mathbb{R}_+ e_i.$$

A function $f\colon \Gamma \to \mathbb{R}$ can be identified to $f = (f_1, \ldots, f_N)$ with $f_i\colon \mathbb{R}_+ \to \mathbb{R}$ and $f(\mathsf{x}) = f_i(x)$ if $\mathsf{x} \in \Gamma_i$.

- The generator of such a process takes the form

$$L_i f(x) = \frac{1}{2}\sigma_i^2(x) f_i''(x) + b_i(x) f_i'(x), \quad x > 0,$$

$$D(L) = \left\{ C^2(\Gamma) \cap C_0(\Gamma) : \; Lf \in C_0(\Gamma) \text{ and } \eta L f(0) = \sum_{i=1}^{N} \rho_i f_i'(0) \right\},$$

where the condition $Lf \in C_0(\Gamma)$ means

$$\lim_{x\to 0} L_i f_i(x) = \lim_{x \to 0} L_j f_j(0) \quad \text{for all } i, j \in \{1, \ldots, N\}.$$

Theorem (Freidlin-Wentzell (1993), Freidlin-Sheu (2000))

The unbounded linear operator $(L, D(L))$ *is the generator of a conservative Feller process* X *on* Γ *having continuous paths. Moreover, for* $\eta = 0$, *writing* $X(t) = (i(t), x(t))$, *we have the following.*

(i)

$$\int_0^t \mathbb{1}_{\{x(s)=0\}}\, ds = 0 \quad \text{for all } t \geq 0,$$

(ii) *there exists a one-dimensional Brownian motion* W *and a continuous increasing process* ℓ_X, *both adapted to the natural filtration of* X, *such that*

$$dx(t) = \sigma_{i(t)}(x(t))dW_t + b_{i(t)}(x(t))dt + d\ell_X(t), \tag{2}$$

and the process ℓ_X *increases only when* $x(t) = 0$. *Furthermore, we have*

$$\left(\sum_{i=1}^{N} \frac{2\rho_i}{\sigma_i^2(0)}\right) \ell_X(t) = \lim_{\delta \to 0} \frac{1}{\delta} \int_0^t \mathbb{1}_{\{x(s)\leq\delta\}}\, ds \quad \text{in expectation.}$$

The Freidlin-Sheu-Itô formula for nonsticky diffusions

Theorem (Freidlin-Sheu (2000))

Let $X = (i(\cdot), x(\cdot))$ be a nonsticky diffusion on Γ (i.e. $\eta = 0$), started at $\mathsf{x} \in \Gamma$, and let $f \in C_b^2(\Gamma)$. Then, almost surely,

$$f(X(t)) - f(\mathsf{x}) = \int_0^t \sigma_{i(s)}(x(s)) f'_{i(s)}(x(s))\, dW(s) + \int_0^t L_{i(s)} f_{i(s)}(x(s))\, ds + \left(\sum_{i=1}^N \rho_i f'_i(0^+) \right) \ell_X(t).$$

Characterization of sticky diffusions

Theorem (B.-Colantoni (2024))

Let Y be a nonsticky diffusion on Γ. Set $V(t) = t + \eta\ell_Y(t)$ and

$$V^{-1}(t) = \inf\{s > 0 : V(s) > t\}.$$

Then

- *the process $X(t) = Y(V^{-1}(t))$ is a sticky diffusion on Γ with stickiness parameter η;*
- *up to an extension of the filtered probability space, there exists a one-dimensional Brownian motion W and an increasing process ℓ_X, that almost surely increases only when $x(t) = 0$, such that, almost surely, we have*

$$\begin{cases} dx(t) = \sigma_{i(t)}(x(t))\mathbb{1}_{\{x(t)\neq 0\}}\, dW_t + b_{i(t)}(x(t))\mathbb{1}_{\{x(t)\neq 0\}}\, dt + d\ell_X(t), \\ \eta\ell_X(t) = \int_0^t \mathbb{1}_{\{x(s)=0\}} ds. \end{cases}$$

Freidlin-Sheu-Itô formula for sticky diffusions

Theorem (B.-Colantoni (2024))

Let $X = Y(V^{-1}(\cdot))$ be a sticky diffusion on Γ. Then, for every $f \in C_b^{1,2}(\mathbb{R}_+ \times \Gamma)$, almost surely, we have

$$f(t, X(t)) = f(0, X(0)) + \int_0^t \left(\partial_t f_{i(s)}(s, x(s)) + L f_{i(s)}(s, x(s))\right) \mathbb{1}_{\{x(s)\neq 0\}} ds + \int_0^t \sigma_{i(s)}(x(s)) \partial_x f_{i(s)}(s, x(s)) \mathbb{1}_{\{x(s)\neq 0\}} dW(s) + \int_0^t \left(\eta\, \partial_t f(s, 0) + \sum_{i=1}^N \rho_i \partial_x f_i(s, 0)\right) d\ell_X(s).$$

If we also assume $f(t, \cdot) \in D(L)$ for all $t \in \mathbb{R}_+$, we also have

$$f(t, X(t)) = f(0, X(0)) + \int_0^t \left(\partial_s f_{i(s)}(s, x(s)) + L f_{i(s)}(s, x(s))\right) ds + \int_0^t \sigma_{i(s)}(x(s)) \partial_x f_{i(s)}(s, x(s)) \mathbb{1}_{\{x(s)\neq 0\}} dW(s) + \int_0^t \left(-\eta\, L f(s, 0) + \sum_{i=1}^N \rho_i \partial_x f_i(s, 0)\right) d\ell_X(s).$$

More general networks

- We consider a network Γ in $\mathbb{R}^d$,
- Vertices are denoted v_i and the set of vertices is $\mathcal{V}$ indexed by $i \in \mathcal{I}$,
- Edges are denoted Γ_α and $\mathcal{E}$ is the set of edges indexed by $\alpha \in \mathcal{A}$.

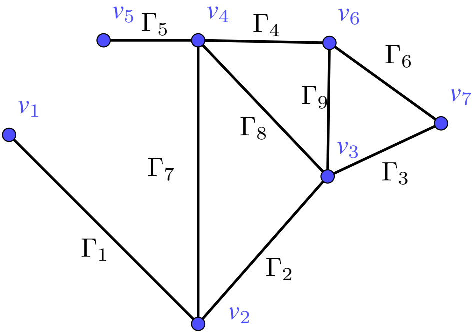

Figure: Example of network.

Stationary distribution for diffusions on networks

- The generator for diffusions on general networks takes the form

$$L_\alpha f(x) = \mu_\alpha \partial^2 f(x) + b_\alpha(x)\partial f(x) \qquad \text{for all } x \in \Gamma_\alpha \setminus \mathcal{V},\ \alpha \in \mathcal{A},$$

$$D(L) = \left\{ f \in \mathscr{C}^2(\Gamma) : \begin{array}{c} Lf \in C(\Gamma),\ \eta_v Lf(v) + \sum_{\alpha\in\mathcal{A}_v} \mu_\alpha \gamma_{v,\alpha}\partial_\alpha f(v) = 0 \\ \text{for all } v \in \mathcal{V} \setminus \partial\mathcal{V}, \partial_\alpha f(v) = 0 \text{ for all } v \in \partial\mathcal{V},\ \alpha \in \mathcal{A}_v \end{array} \right\}.$$

- We recall that a stationary distribution for the processes generated by $(L, D(L))$ is a measure $\mathfrak{m} \in \mathcal{P}(\Gamma)$ such that

$$\int Lf(x)\mathfrak{m}(dx) = 0 \quad \text{for all } f \in D(L).$$

Characterization of stationary distributions

Proposition (Achdou-Dao-Ley-Tchou (2019), B.-Camilli (2024))

The measure defined by

$$\mathfrak{m} = m\,\mathscr{L} + \sum_{v \in \mathcal{V} \setminus \partial\mathcal{V}} \eta_v T_v[m] \delta_v,$$

where m is a weak solution to

$$\begin{cases} -\mu_\alpha \partial^2 m(x) - \partial\left(b(x)m(x)\right) = 0 & \text{for all } x \in \Gamma_\alpha,\ \alpha \in \mathcal{A}, \\ \frac{m_{|\Gamma_\alpha}(v)}{\gamma_{v,\alpha}} = \frac{m_{|\Gamma_\beta}(v)}{\gamma_{v,\beta}} =: T_v[m] & \text{for all } \alpha, \beta \in \mathcal{A}_v,\ v \in \mathcal{V} \setminus \partial\mathcal{V}, \\ \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \partial_\alpha m_{|\Gamma_\alpha}(v) + n_{v,\alpha} m_{|\Gamma_\alpha}(v) b_{|\Gamma_\alpha}(v) = 0 & \text{for all } v \in \mathcal{V}, \\ m \geq 0,\ 1 \geq \int_\Gamma m\,dx = 1 - \sum_{v \in \mathcal{V} \setminus \partial\mathcal{V}} \eta_v T_v[m] \geq 0. \end{cases} \tag{3}$$

is a stationary distribution for the Markov process generated by $(L, D(L))$. Moreover, there exists a unique weak solution to (3).

An infinite horizon optimal control problem

- We fix a set of admissible controls

$$\mathfrak{A} = \{a : \Gamma \to \mathbb{R} :\ a \in \mathcal{PC}(\Gamma),\ \|a\|_{\mathcal{PC}(\Gamma)} \le R\},$$

and, for each $a \in \mathfrak{A}$, we consider the Markov process X^a with generator

$$L^a_\alpha f(x) = \mu_\alpha \partial^2 f(x) + b_\alpha(x, a_\alpha(x))\partial f(x) \qquad \text{for all } x \in \Gamma_\alpha \setminus \mathcal{V},\ \alpha \in \mathcal{A}.$$

- We consider the cost functional

$$J(x,a) = \mathbb{E}_x\left[\int_0^{+\infty} e^{-\lambda s}\Big[\Big(\ell(X^a(s), a(X^a(s))) + F(X^a(s))\Big)\mathbb{1}_{\{X(s)\notin\mathcal{V}\}} + \sum_{v\in\mathcal{V}\setminus\partial\mathcal{V}} \theta_v \mathbb{1}_{\{X^a(s)=v\}}\Big]\, ds\right]$$

and define the value function

$$V(x) = \inf_{a\in\mathfrak{A}} J(x,a).$$

HJB equation

- We expect the value function V to be a solution to the HJB equation

$$\begin{cases} -\mu_\alpha \partial^2 u(x) + H(x, \partial u(x)) + \lambda u(x) = F(x) & \text{for all } x \in \Gamma_\alpha \setminus \mathcal{V},\ \alpha \in \mathcal{A}, \\ u_{|\Gamma_\alpha}(v) = u_{|\Gamma_\beta}(v) & \text{for all } \alpha, \beta \in \mathcal{A}_v,\ v \in \mathcal{V}, \\ \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v,\alpha} \partial_\alpha u(v) = \eta_v(\theta_v - \lambda u(v)) & \text{for all } v \in \mathcal{V} \setminus \partial\mathcal{V}, \\ \partial_\alpha u(v) = 0 & \text{for all } v \in \partial\mathcal{V}, \end{cases} \tag{4}$$

where

$$H_\alpha(x,p) := \sup_{|a| \le R} \left\{ -b_\alpha(x,a)p - \ell_\alpha(x,a) \right\}, \qquad \alpha \in \mathcal{A}.$$

- The difficult point is to justify the transmission condition.
- Assuming we know a classical solution to (4), one option is to prove a verification theorem. Such results were proved by Fleming and Soner under the restrictive assumption

$$u \in \bigcap_{a \in \mathfrak{A}} D(L^a).$$

Verification theorem

- The Fleming-Soner verification theorem is applicable to the case of nonsticky diffusions but requires very strong assumptions on the optimal control problem in the case of sticky diffusions. This comes from the fact that the transmission condition depends directly on the control a.
- The Freidlin-Sheu-Itô formula for sticky diffusions then allows to prove the following.

Proposition (Verification Theorem, B.-Camilli (2024))

Then, for every $a \in \mathfrak{A}$, we have $u(x) \leq J(x,a)$. In particular we have $u(x) \leq V(x)$. Moreover, if, for each $(x,p) \in \Gamma_\alpha \times \mathbb{R}$, there exists a unique $a_\alpha^\star = a_\alpha^\star(x,p)$, with $|\alpha_\alpha^\star| \leq R$ for every $\alpha \in \mathcal{A}$, such that

$$H_\alpha(x,p) = -b_\alpha(x, a_\alpha^\star)p - \ell_\alpha(x, a_\alpha^\star)$$

and the map $a_\alpha^\star : \Gamma_\alpha \times \mathbb{R} \to \mathbb{R}$ is continuous. Then $u(x) \geq V(x)$ for all $x \in \Gamma$.

Well-posedness of the discounted HJB equation

Proposition (Achdou-Dao-Ley-Tchou (2019), B.-Camilli (2024))

Assume that

- *There exists a constant $C_H > 0$ such that*

$$|H_\alpha(x,p)| \le C_H \left(1 + |p|^2\right) \quad \text{for all } (x,p) \in \Gamma_\alpha \times \mathbb{R}, \tag{5}$$

- *F belongs to $PC^\varsigma(\Gamma)$ for some $\varsigma \in (0,1)$.*

Then, there exists a unique solution classical solution to

$$\begin{cases} -\mu_\alpha \partial^2 u(x) + H(x, \partial u(x)) + \lambda u(x) = F(x) & \text{for all } x \in \Gamma_\alpha \setminus \mathcal{V},\ \alpha \in \mathcal{A}, \\ u_{|\Gamma_\alpha}(v) = u_{|\Gamma_\beta}(v) & \text{for all } \alpha, \beta \in \mathcal{A}_v,\ v \in \mathcal{V}, \\ \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v,\alpha} \partial_\alpha u(v) = \eta_v(\theta_v - \lambda u(v)) & \text{for all } v \in \mathcal{V} \setminus \partial\mathcal{V}, \\ \partial_\alpha u(v) = 0 & \text{for all } v \in \partial\mathcal{V}. \end{cases}$$

The ergodic problem

Theorem (Achdou-Dao-Ley-Tchou (2019), B.-Camilli (2024))

Under assumptions

- *There exists a constant* $C_H > 0$ *and* $q \in (1,2]$ *such that*

$$|H_\alpha(x,p)| \leq C_H\left(1+|p|^q\right) \quad \text{for all } (x,p) \in \Gamma_\alpha \times \mathbb{R}, \tag{6}$$

$$H_\alpha(x,p) \geq C_H^{-1}|p|^q - C_H \quad \text{for all } (x,p) \in \Gamma_\alpha \times \mathbb{R}; \tag{7}$$

- F *belongs to* $PC^\varsigma(\Gamma)$ *for some* $\varsigma \in (0,1)$*;*

there exists a unique classical solution (u,ρ) *to*

$$\begin{cases} -\mu_\alpha \partial^2 u(x) + H(x, \partial u(x)) + \rho = F(x) & \text{for all } x \in \Gamma_\alpha \setminus \mathcal{V},\ \alpha \in \mathcal{A}, \\ u_{|\Gamma_\alpha}(v) = u_{|\Gamma_\beta}(v) & \text{for all } \alpha, \beta \in \mathcal{A}_v,\ v \in \mathcal{V}, \\ \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v,\alpha} \partial_\alpha u(v) = \eta_v(\theta_v - \rho) & \text{for all } v \in \mathcal{V} \setminus \partial\mathcal{V}, \\ \partial_\alpha u(v) = 0 & \text{for all } v \in \partial\mathcal{V}, \\ \int_\Gamma u(x)dx = 0. & \end{cases}$$

The MFG system

The Mean Field Game system is

$$
\begin{cases}
(i)\ -\mu_\alpha \partial^2 u(x) + H(x, \partial u(x)) + \rho = F[\mathfrak{m}] & \text{for all } x \in \Gamma_\alpha \setminus \mathcal{V},\ \alpha \in \mathcal{A}, \\
u_{|\Gamma_\alpha}(v) = u_{|\Gamma_\beta}(v) & \text{for all } \alpha, \beta \in \mathcal{A}_v,\ v \in \mathcal{V}, \\
\sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v,\alpha} \partial_\alpha u(v) = \eta_v \left(\theta_v + F[\mathfrak{m}](v) - \rho\right) & \text{for all } v \in \mathcal{V} \setminus \partial \mathcal{V}, \\
\partial_\alpha u(v) = 0 & \text{for all } v \in \partial \mathcal{V}, \\
\int_\Gamma u \, dx = 0 & \\
(ii)\ \mathfrak{m} = m \mathscr{L} + \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v[m] \delta_v, & \\
-\mu_\alpha \partial^2 m(x) - \partial \left(\partial_p H(x, \partial u(x)) m(x)\right) = 0 & \text{for all } x \in \Gamma_\alpha,\ \alpha \in \mathcal{A}, \\
\frac{m_{|\Gamma_\alpha}(v)}{\gamma_{v,\alpha}} = \frac{m_{|\Gamma_\beta}(v)}{\gamma_{v,\beta}} =: T_v[m] & \text{for all } \alpha, \beta \in \mathcal{A}_v,\ v \in \mathcal{V} \setminus \partial \mathcal{V}, \\
\sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \partial_\alpha m_{|\Gamma_\alpha}(v) + n_{v,\alpha} m_{|\Gamma_\alpha}(v) \partial_p H_\alpha(v, \partial u_{|\Gamma_\alpha}(v)) = 0 & \forall v \in \mathcal{V}, \\
m \geq 0,\ 1 \geq \int_\Gamma m \, dx = 1 - \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v[m] \geq 0. &
\end{cases}
$$

Well-posedness of the MFG system

Theorem (Achdou-Dao-Ley-Tchou (2019), B.-Camilli(2024))

Assume that

- *There exists a constant* $C_H > 0$ *and* $q \in (1, 2]$ *such that*

$$|H_\alpha(x,p)| \leq C_H \left(1 + |p|^q\right) \quad \text{for all } (x,p) \in \Gamma_\alpha \times \mathbb{R},$$
$$|\partial_p H_\alpha(x,p)| \leq C_H \left(1 + |p|^{q-1}\right) \quad \text{for all } (x,p) \in \Gamma_\alpha \times \mathbb{R},$$
$$H_\alpha(x,p) \geq C_H^{-1} |p|^q - C_H \quad \text{for all } (x,p) \in \Gamma_\alpha \times \mathbb{R}.$$

- $F \colon \mathcal{P}_1(\Gamma) \to PC^\varsigma(\Gamma)$ *is continuous and takes values in a bounded subset of* $PC^\varsigma(\Gamma)$.

Then there exists a solution (u, ρ, m) *to the MFG system. Moreover, if*

- *the map* $p \mapsto H(x,p)$ *is (strictly) convex and* F *satisfies*

$$\int_\Gamma (F[\mathfrak{m}_1] - F[\mathfrak{m}_2])(\mathfrak{m}_1 - \mathfrak{m}_2)(dx) > 0 \, (\geq 0),$$

then there is at most one solution.

Thank you for your attention !