Approximation of an optimal control problem posed on a network with a perturbed problem in the whole space

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• A lot of recent developments on optimal control problems and HJB equations on networks or stratified structures :

Achdou-Camilli-Cutrì-Tchou (2013), Imbert-Monneau-Zidani (2013), Camilli-Schieborn (2013), Imbert-Monneau (2017), Barles-Briani-Chasseigne (2014), Achdou-Oudet-Tchou (2015), Lions-Souganidis (2016), Graber-Hermosilla-Zidani (2017), Carlini-Festa-Forcadel (2020), Fayad-Forcadel-Ibrahim (2022, Siconolfi (2022), $\cdots$

and, for a recent overview, Barles-Chasseigne (2024), $\cdots$

• A subject beyond the classical theory

• A new theory has been designed for issues on networks by, first considering trajectories and costs adapted to networks

A different approach

Perturbing a classical optimal control problem in $\mathbb{R}^2$, with a singular term pushing the trajectories towards the network

- Goals :

(1) Link between the limit problem and some known optimal control problems

(2) Encode the geometry of the network in the perturbation

- A related work by Achdou-Tchou (2015) exists where the control problem on a junction is approximated by a state constraint problem in an ε-neighborhood of the junction

A classical optimal control problem in $\mathbb{R}^2$

Assumptions :

A : compact subset in $\mathbb{R}^2$ (the controls)

$f : \mathbb{R}^2 \times A \to \mathbb{R}^2$, $\ell : \mathbb{R}^2 \times A \to \mathbb{R}$: continuous s.t., for all $a \in A$, $x, y \in \mathbb{R}^2$,

$$|f(x,a)|, |\ell(x,a)| \leq M, \quad |f(x,a) - f(y,a)|, |\ell(x,a) - \ell(y,a)| \leq M|x-y|$$

Consider the infinite horizon control problem

$$V(x) := \inf_{\alpha \in L^\infty((0,\infty);A)} \int_0^\infty e^{-\lambda t} \ell(X^{x,\alpha}(t), \alpha(t)) dt,$$

where

$$X^{x,\alpha}(0) = x \in \mathbb{R}^2, \quad \dot{X}^{x,\alpha}(t) = f(X^{x,\alpha}(t), \alpha(t)), \ t > 0.$$

• Classical way : restrict the class of controls to those leading to $X^{x,\alpha}(\cdot)$ on Γ

$$\mathcal{A}_x := \{\alpha \in L^\infty((0,\infty);A) : X^{x,\alpha}(t) \in \Gamma \text{ for all } t \geq 0\}$$

• Under some additional assumptions on the set of controls,

$$V_\Gamma(x) := \inf_{\alpha \in \mathcal{A}_x} \int_0^\infty e^{-\lambda t} \ell(X^{x,\alpha}(t), \alpha(t)) dt$$

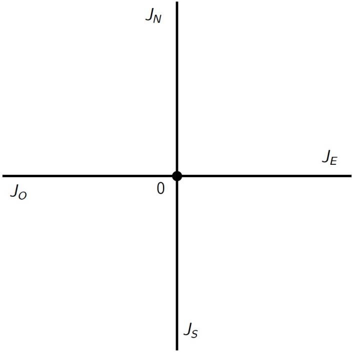
unique viscosity solution to a HJB equation on Γ (Achdou-Camilli-Cutrì-Tchou (2013) and Achdou-Oudet-Tchou (2015))

• A notion of solution, in many usual cases, equivalent to those developed in Imbert-Monneau-Zidani (2013).

Consider the very simple junction

$$\Gamma = \{O\} \cup \bigcup_{i\in\{E,N,W,S\}} (0,\infty)e_i,$$

where $e_E = (1,0)$, $e_N = (0,1)$, $e_W = (-1,0)$ and $e_S = (0,-1)$.

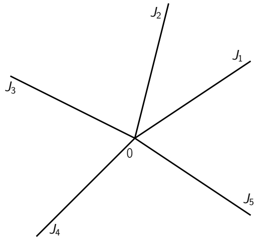


$\Gamma = \{d = 0\} = \{\nabla d = (0,0)\}$, where $d(x) = x_1^2 x_2^2$, $x = (x_1, x_2) \in \mathbb{R}^2$

- With such Γ and d computations are "simple"

- Our framework should be generalized to junctions with a finite number of branches

$$\Gamma = \{O\} \cup \bigcup_{1 \le i \le \ell} (0, \infty) e_i$$



A $\frac{1}{\varepsilon}$-perturbed optimal control problem

Consider

$$V^\varepsilon(x) := \inf_{\alpha \in L^\infty((0,\infty);A)} \int_0^\infty e^{-\lambda t} \ell(X^{x,\alpha,\varepsilon}(t), \alpha(t))dt,$$

where $X^{x,\alpha,\varepsilon}$ is the solution to

$$(S_\varepsilon) \begin{cases} \dot{X}^{x,\alpha,\varepsilon}(t) = F^\varepsilon(X^{x,\alpha,\varepsilon}(t), \alpha(t)) := f(X^{x,\alpha,\varepsilon}(t), \alpha(t)) - \frac{1}{\varepsilon} \nabla d(X^{x,\alpha,\varepsilon}(t)), t > 0 \\ X^{x,\alpha,\varepsilon}(0) = x, \end{cases}$$

and set

$$k^{x,\alpha,\varepsilon}(t) = \frac{1}{\varepsilon} \int_0^t \nabla d(X^{x,\alpha,\varepsilon}(s))ds, \quad t \geq 0.$$

Gradient trajectories

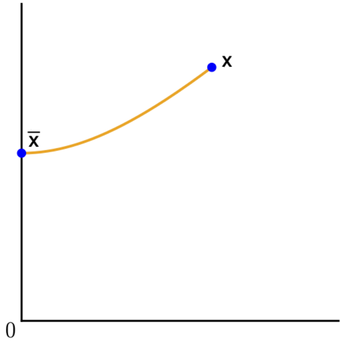
Consider

$$\begin{cases} \dot{X}(t) = -\nabla d(X(t)), & t > 0, \\ X(0) = x \in \mathbb{R}^2. \end{cases}$$

Theorem (Łojasiewicz (1984))

For any $x \in \mathbb{R}^2$, the limit of X, as $t \to \infty$, exists and belongs to Γ

$$\overline{x} := \lim_{t\to\infty} X(t) \in \Gamma$$



Gradient trajectories

Theorem

Take $d_\Gamma \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^2;[0,\infty))$ *s.t.* $\Gamma = \{d_\Gamma = 0\} = \{\nabla d_\Gamma = (0,0)\}$

and there exists $\nu > 0$, $\theta \in (0,1)$ *such that*

$$|\nabla d_\Gamma(x)| \geq \nu d_\Gamma(x)^\theta \text{ for all } x \in \mathbb{R}^2 \quad \textit{(Łojasiewicz inequality)}$$

Then, $\forall\, x \in \mathbb{R}^2$, *the unique solution* $Z^x : [0,\infty) \to \mathbb{R}^2$ *of*

$$\dot{X} = -\nabla d_\Gamma(X),\ X(0) = x$$

has a limit $\overline{x} \in \Gamma$, *as* $t \to \infty$. *Moreover, the map*

$$\phi_{d_\Gamma} : x \in \mathbb{R}^2 \mapsto \overline{x} := \lim_{t\to\infty} Z^x(t) \in \Gamma$$

is continuous.

Gradient trajectories

Proposition

For $d_\Gamma(x) = x_1^2 x_2^2$, $Z^x(t)$ *lies on a hyperbola* $\perp$ *to the level sets of* d_Γ *:*

$$(Z_2^x(t))^2 - (Z_1^x(t))^2 = x_2^2 - x_1^2 \quad \textit{for all } t \geq 0,$$

and

$$\phi_{d_\Gamma}(x) = \begin{cases} \operatorname{sgn}(x_2)\sqrt{x_2^2 - x_1^2}\, e_N & \textit{if } |x_2| \geq |x_1|, \\ \operatorname{sgn}(x_1)\sqrt{x_1^2 - x_2^2}\, e_E & \textit{if } |x_1| \geq |x_2|, \end{cases}$$

is $\frac{1}{2}$*-Hölder continuous on* $\mathbb{R}^2$

Theorem (Convergence of trajectories)

Take $x \in \mathbb{R}^2$ *and* $\alpha \in L^\infty((0,\infty); A)$*. Then,*

1. (S_ε) *admits a unique solution* $X^{x,\alpha,\varepsilon} \in W^{1,\infty}_{\mathrm{loc}}([0,\infty))$
2. *Up to extraction,* $(X^{x,\alpha,\varepsilon}, k^{x,\alpha,\varepsilon})$ *converges locally uniformly on* $(0,\infty)$ *to some* $(X^{x,\alpha}, k^{x,\alpha}) \in H^1_{\mathrm{loc}}(0,\infty) \times H^1_{\mathrm{loc}}(0,\infty)$ *when* $\varepsilon \to 0$*, s.t.*
   - $\forall t \geq 0, \quad X^{x,\alpha}(t) = x + \int_0^t f(X^{x,\alpha}(s), \alpha(s))ds - k^{x,\alpha}(t),$
   - $\forall t > 0, \quad X^{x,\alpha}(t) \in \Gamma,$
   - $X^{x,\alpha}(0^+) = \overline{x}, \; k^{x,\alpha}(0^+) = x - \overline{x},$
3. *If* $x \in \Gamma$*, then, up to extraction,* $(X^{x,\alpha,\varepsilon}, k^{x,\alpha,\varepsilon})$ *converges locally uniformly on* $[0,\infty)$ *to* $(X^{x,\alpha}, k^{x,\alpha}) \in H^1_{\mathrm{loc}}[0,\infty) \times H^1_{\mathrm{loc}}[0,\infty)$ *as* $\varepsilon \to 0$*.*
4. $(X^{x,\alpha}, k^{x,\alpha}) \in W^{1,\infty}_{\mathrm{loc}}(0,\infty) \times W^{1,\infty}_{\mathrm{loc}}(0,\infty)$ *and*

$$|\dot{X}^{x,\alpha}(t)|, |\dot{k}^{x,\alpha}(t)| \leq |f|_\infty \quad \text{a.e. } t \in (0,\infty).$$

Set

$$\omega(x,\alpha) = \{X^{x,\alpha} \in H^1_{\mathrm{loc}}(0,\infty) : (X^{x,\alpha}, k^{x,\alpha}) \text{ is a subsequential limit of } (X^{x,\alpha,\varepsilon}, k^{x,\alpha,\varepsilon})\}.$$

- $\omega(x,\alpha) \neq \emptyset$

- Does $\omega(x,\alpha)$ reduces to one element ?

- We cannot characterize completely the solution $(X^{x,\alpha}, k^{x,\alpha})$ of the limit problem.

- $(X^{x,\alpha}, k^{x,\alpha})$ is a solution to a Skorokhod problem

- Such a problem on Γ faces nonuniqueness and instability

A partial characterization result inside the edges

Take $X^{x,\alpha} \in \omega(x,\alpha)$. By continuity of the trajectory for $t > 0$,

$$(0,\infty) = \mathcal{Z}(X^{x,\alpha}) \cup \bigcup_{0 \le n \le \ell} (a_n, b_n),$$

- $\mathcal{Z}(X^{x,\alpha}) := \{t \in (0,\infty) : X^{x,\alpha}(t) = O\}$: times when the trajectory is at O
- (a_n, b_n) : time intervals when the trajectory is inside a branch.
- $\ell = \ell(X^{x,\alpha})$ is possibly ∞

Theorem (Dynamic inside the branches : $\Gamma \setminus \{O\}$)

Take $X^{x,\alpha} \in \omega(x,\alpha)$.

1. *If* $a_n = 0$, *then, for all* $t \in [0, b_n]$,

$$\begin{cases} X^{x,\alpha}(t) = \overline{x} + \displaystyle\int_0^t \langle f(X^{x,\alpha}(s), \alpha(s)), e_{i(n)} \rangle e_{i(n)} ds, \\ k^{x,\alpha}(t) = x - \overline{x} + \displaystyle\int_0^t \langle f(X^{x,\alpha}(s), \alpha(s)), e_{i(n)}^{\perp} \rangle e_{i(n)}^{\perp} ds, \end{cases}$$

2. *If* $a_n \neq 0$, *then, for all* $t \in [a_n, b_n]$,

$$\begin{cases} X^{x,\alpha}(t) = \displaystyle\int_{a_n}^t \langle f(X^{x,\alpha}(s), \alpha(s)), e_{i(n)} \rangle e_{i(n)} ds, \\ k^{x,\alpha}(t) = k^{x,\alpha}(a_n) + \displaystyle\int_{a_n}^t \langle f(X^{x,\alpha}(s), \alpha(s)), e_{i(n)}^{\perp} \rangle e_{i(n)}^{\perp} ds. \end{cases}$$

▷ Characterization of $(X^{x,\alpha}, k^{x,\alpha})$ inside the edges (in terms of projections of the dynamic f)

▷ $(X^{x,\alpha}, k^{x,\alpha})$ uniquely determined once $(a_n)_n$ and $(b_n)_n$ known : hard to obtain for general f.

- A "Zeno effect" example, where $X^{O,\alpha}$ visits all the edges for arbitrary small t

Behavior of $X^{O,\alpha}$ with locally constant control near O

Once at O, it's difficult to say what happens later

- How long does the trajectory remain at O ?

- In which branch does it enter ? $\cdots$

Lemma (A trajectory visiting all branches instantaneously)

There exists a control $\alpha \in \mathcal{A}_O$ such that $\omega(O, \alpha) = \{X^{O,\alpha}\}$ and, for all $\delta > 0$ and $i \in \{E, N, W, S\}$, there exists $t \in (0, \delta)$ such that $X^{O,\alpha}(t) \in (0, \infty)e_i$.

- With such α one cannot determine the branch the trajectory enters

Convergence to the control problem on the network

Consider the perturbed optimal control problem

$$V^{\varepsilon}(x) := \inf_{\alpha \in L^{\infty}((0,\infty);A)} \int_0^{\infty} e^{-\lambda t} \ell(X^{x,\alpha,\varepsilon}(t), \alpha(t)) dt, \quad x \in \mathbb{R}^2$$

and the natural asymptotic optimal control problem on Γ

$$\overline{V}(x) := \inf_{\substack{\alpha \in L^{\infty}((0,\infty);A) \\ X^{x,\alpha} \in \omega(x,\alpha)}} \int_0^{\infty} e^{-\lambda t} \ell(X^{x,\alpha}(t), \alpha(t)) dt, \quad x \in \mathbb{R}^2.$$

Convergence to the control problem on the network

Two additional assumptions

- There exists $\eta > 0$ such that, for every $x \in \mathbb{R}^2$,

$$\overline{B}(0,\eta) \subset f(x,A)$$

- For every $x \in \mathbb{R}^2$, $a \in A$,

$$\ell(x,a) = \ell(x)$$

Theorem

1. *V^ε converges locally uniformly on $\mathbb{R}^2$ to $\overline{V}$*
2. $\overline{V}(x) = V_\Gamma(\overline{x}), \quad \forall x \in \mathbb{R}^2$

Proof : by tracking the trajectories

Some remarks

• Complete answer to our problem. But, we need the assumptions

$$\overline{B}(0,\eta) \subset f(x,A) \quad \text{for some} \quad \eta > 0$$

and

$$\ell(x,a) = \ell(x), \quad \forall\,(x,a) \in \mathbb{R}^2 \times A$$

• In the literature on HJB equations on networks, costs may **depend directly on the control** but are designed for controls that **keep the trajectory on** Γ

• Here, we allow any control

Some estimates independent of α and $\varepsilon > 0$

Lemma (Well-posedness of (S_ε) for $\varepsilon > 0$)

There exists a unique solution $X^{x,\alpha,\varepsilon} \in W^{1,\infty}_{\mathrm{loc}}([0,\infty))$ *of* (S_ε) *and*

$$|X^{x,\alpha,\varepsilon}(t)| \leq |x| + |f|_\infty t, \quad \forall\, t \geq 0.$$

Lemma

There exists $C = C(T, |f|_\infty, |x|) > 0$ *s.t., for any* $0 \leq t_1 \leq t_2 \leq T$*,*

$$\frac{1}{\varepsilon}\int_{t_1}^{t_2} d(X^{x,\alpha,\varepsilon}(t))dt \leq C(t_2 - t_1) + \frac{1}{4}\left(d^{\frac{1}{2}}(X^{x,\alpha,\varepsilon}(t_1)) - d^{\frac{1}{2}}(X^{x,\alpha,\varepsilon}(t_2))\right).$$

Invariant subsets for (S_ε)

- $S \subset \mathbb{R}^2$ is invariant for (S_ε) if, for any $t \geq 0$, $X^{x,\alpha,\varepsilon}(t) \in S$.

- A sufficient condition, for a subset S with C^1 boundary, to be invariant is

$$\langle F^\varepsilon(x,a), n_{\partial S}(x)\rangle \leq 0, \quad \text{for all } x \in \partial S,\ a \in A,$$

where $n_{\partial S}(x)$ = the outward unit normal vector to ∂S at x.

- For $\lambda > 0$, define the subset

$$Z(\lambda) := \{x \in \mathbb{R}^2 : d(x) \leq \lambda\}.$$

- Entry time in $Z(\lambda)$ from an initial position $x \in \mathbb{R}^2$:

$$t^{x,\alpha,\varepsilon}(\lambda) := \begin{cases} \inf\{t \geq 0 : X^{x,\alpha,\varepsilon}(t) \in Z(\lambda)\}, & \text{if } X^{x,\alpha,\varepsilon}(\cdot) \text{ reaches } Z(\lambda), \\ \infty, & \text{otherwise.} \end{cases}$$

Proposition (Invariant subsets and entry times)

1. $$d(X^{x,\alpha,\varepsilon}(t_2)) - d(X^{x,\alpha,\varepsilon}(t_1)) \le (t_2 - t_1)\frac{|f|_\infty^2}{4}\varepsilon, \quad \forall\, 0 \le t_1 \le t_2$$

2. $\lambda \ge \lambda_\varepsilon := \frac{1}{4}|f|_\infty^{4/3}\varepsilon^{4/3} \implies Z(\lambda)$ *is invariant for* (S_ε) *and*

$$d(X^{x,\alpha,\varepsilon}(t)) \le \frac{1}{4}|f|_\infty^{4/3}\varepsilon^{4/3}, \quad \textit{for all } x \in \Gamma \textit{ and } t \ge 0.$$

$$d(x) > \lambda_\varepsilon \implies d(X^{x,\alpha,\varepsilon}(t_2)) \le d(X^{x,\alpha,\varepsilon}(t_1)),\ \forall\, 0 \le t_1 \le t_2 \le t^{x,\alpha,\varepsilon}(\lambda_\varepsilon)$$

3. *For any* $\gamma < 1$ *and* ε *small enough, we have* $t^{x,\alpha,\varepsilon}(\varepsilon^\gamma) \le \frac{1}{2}\sqrt{d(x)}\varepsilon^{1-\gamma}$

4. *For any* $\lambda > 0$ *and* $T > 0$, *there exists* $C = C(T, |f|_\infty, |x|)$ *such that*

$$t^{x,\alpha,\varepsilon}(\lambda) \le C\frac{\varepsilon}{\lambda}$$

Qualitative properties of the trajectories $X^{x,\alpha}$

Behavior of the trajectories passing through O with constant controls

Take $e_\theta = (\cos\theta, \sin\theta)$ and consider a trajectory $X^{x,\alpha}$ such that

$$X^{x,\alpha}(t_0) = O, \text{ for some } \; t_0 \geq 0,$$

and $\alpha(t) \equiv e_\theta$ in some interval $[t_0 - \tau, t_0 + \tau]$.

- If $x \neq O$, scaling and permutation arguments allow to assume $x = (0,1) = e_N$

Lemma (Scaling property)

$$X^{x,e_\theta,\varepsilon}(t) = \frac{1}{\rho} X^{\rho x, e_\theta, \rho^3 \varepsilon}(\rho t), \quad \forall\, \rho > 0$$

$$(S_\varepsilon) \quad \begin{cases} \dot{X}_1^\varepsilon(t) &= \cos\theta - \frac{2}{\varepsilon} X_1^\varepsilon(t)(X_2^\varepsilon(t))^2, \\ \dot{X}_2^\varepsilon(t) &= \sin\theta - \frac{2}{\varepsilon}(X_1^\varepsilon(t))^2 X_2^\varepsilon(t), \\ X(0) &= x. \end{cases} \qquad t > 0.$$

Proposition (Trajectories starting from O)

Let $\alpha(t) \equiv e_\theta$ *for* $\theta \in [0, 2\pi]$ *and* $x = O$*. Then,* $\omega(O, e_\theta) = \{X^{O,e_\theta}\}$ *and :*

1. (Non bisector case, $\theta \notin \{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\}$)

   If $i \in \{E, N, W, S\}$ *s.t.* $\langle e_\theta, e_i \rangle > \max_{j \neq i} \langle e_\theta, e_j \rangle$*, then,* X^{O,e_θ}

   enters the branch i *:*

   $$X^{O,e_\theta}(t) = \langle e_\theta, e_i \rangle t e_i, \quad \forall t \geq 0.$$

2. (Bisector case, $\theta \in \{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\}$) *If* $i \neq j$

   s.t. $\langle e_\theta, e_i \rangle = \langle e_\theta, e_j \rangle$*, then,* X^{O,e_θ} *remains at* O *:*

   $$X^{O,e_\theta}(t) = O, \quad \forall t \geq 0.$$

Proposition (Trajectories starting inside a branch)

Let $\alpha(t) \equiv e_\theta$ *for* $\theta \in [0, 2\pi]$ *and* $x = e_N$. *Then,* $\omega(e_N, e_\theta) = \{X^{e_N, e_\theta}\}$ *and :*

1. $\theta \in [0, \pi] \implies X^{e_N, e_\theta}$ *never reaches* O.

2. $\theta \in (\pi, \frac{5\pi}{4}] \implies X^{e_N, e_\theta}$ *enters the branch* W *after passing through* O *:*

$$X^{e_N, e_\theta}(t) = \begin{cases} \big(0, 1 + (\sin\theta)t\big) & \text{if } \; t \in [0, (-\sin\theta)^{-1}], \\ \big((\cos\theta)t, 0\big) & \text{if } \; t \in [(-\sin\theta)^{-1}, \infty). \end{cases}$$

3. $\theta \in (\frac{5\pi}{4}, \frac{7\pi}{4}) \implies X^{e_N, e_\theta}$ *continues into the branch* S *after passing through* O *:*

$$X^{e_N, e_\theta}(t) = \big(0, 1 + (\sin\theta)t\big) \quad \forall t \geq 0.$$

4. $\theta \in [\frac{7\pi}{4}, 2\pi) \implies X^{e_N, e_\theta}$ *enters the branch* E *after passing through* O.

Some elements of the proof

$X^{e_N,e_\theta}(t) = (1 + (\sin\theta)t)e_N$ for all $t \in [0, \underline{t} := \inf\{s \geq 0 : X^{e_N,e_\theta}(s) = O\}]$

Keystone to obtain the result :

prove that, for $t > (-\sin\theta)^{-1}$, $X_1^{e_N,e_\theta,\varepsilon}(t) \leq -\eta$ (resp. $X_2^{e_N,e_\theta,\varepsilon}(t) \leq -\eta$) for some $\eta > 0$ independent of ε.

Lemma

Let $\theta \in (\pi, \frac{3\pi}{2}]$. There exists $t_\varepsilon \in (0, (-\sin\theta)^{-1}]$ such that

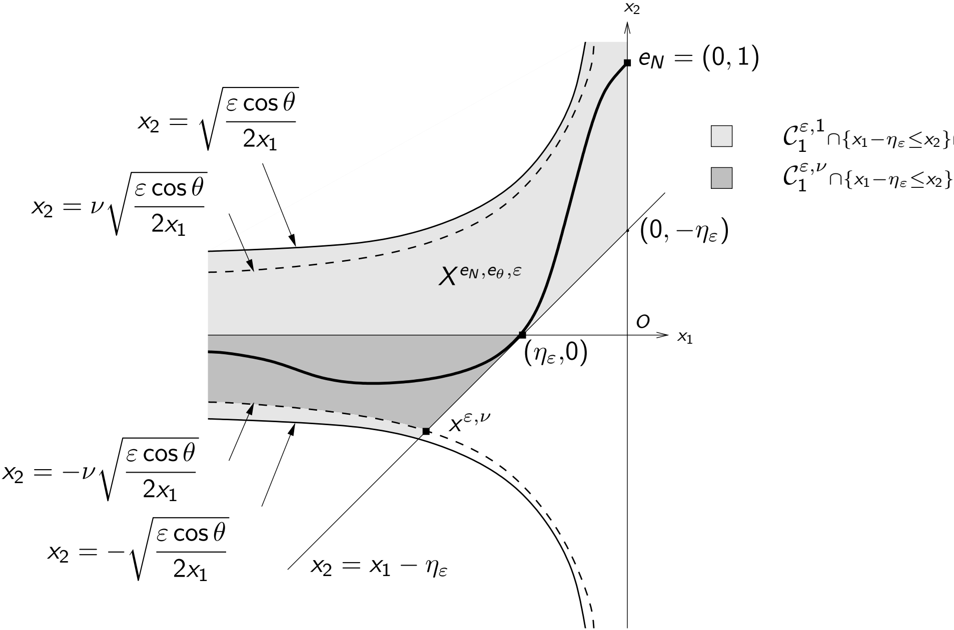
$$\eta_\varepsilon := X_1^{e_N,e_\theta,\varepsilon}(t_\varepsilon) \leq 0 \quad \textit{and} \quad X_2^{e_N,e_\theta,\varepsilon}(t_\varepsilon) = 0,$$

and, for all $t \geq t_\varepsilon$, $X^{e_N,e_\theta,\varepsilon}(t) \in \{x_1 \leq 0\} \cap \{x_2 \leq 0\}$

Lemma

Let $\theta \in (\pi, \frac{3\pi}{2}]$. There exists $\gamma > 0$ (independent of ε) such that

$$-\frac{3}{2}\frac{(-\cos\theta)}{(-\sin\theta)^{2/3}}\varepsilon^{1/3} \leq \eta_\varepsilon = X_1^{e_N,e_\theta,\varepsilon}(t_\varepsilon) \leq -\gamma\varepsilon^{1/3}$$



x_2
$e_N = (0,1)$
$x_2 = \sqrt{\dfrac{\varepsilon\cos\theta}{2x_1}}$
$x_2 = \nu\sqrt{\dfrac{\varepsilon\cos\theta}{2x_1}}$
$(0, -\eta_\varepsilon)$
$X^{e_N, e_\theta, \varepsilon}$
O
x_1
$(\eta_\varepsilon, 0)$
$x^{\varepsilon,\nu}$
$x_2 = -\nu\sqrt{\dfrac{\varepsilon\cos\theta}{2x_1}}$
$x_2 = -\sqrt{\dfrac{\varepsilon\cos\theta}{2x_1}}$
$x_2 = x_1 - \eta_\varepsilon$
$\mathcal{C}_1^{\varepsilon,1} \cap \{x_1 - \eta_\varepsilon \leq x_2\}$
$\mathcal{C}_1^{\varepsilon,\nu} \cap \{x_1 - \eta_\varepsilon \leq x_2\}$

Lemma

Let $\theta \in (\frac{5\pi}{4}, \frac{3\pi}{2}]$. *For* $0 < \nu \leq 1$, *define*

$$\mathcal{C}_2^{\varepsilon,\nu} := \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : F_2^{\varepsilon}(x, e_\theta) = \sin\theta - \frac{2}{\varepsilon} x_1^2 x_2 \leq (1-\nu)\sin\theta \right\}.$$

There exist $0 < \nu < 1$ *sufficiently close to 1, and* $\rho > 0$ *small enough, independent of* ε, *such that the set*

$$\mathcal{C}_2^{\varepsilon,\nu} \cap \{x_2 \leq x_1 + \rho\varepsilon^{1/3}\} \cap \{x_1 \leq 0\} \cap \{x_2 \leq 0\}$$

is invariant for (S_ε).

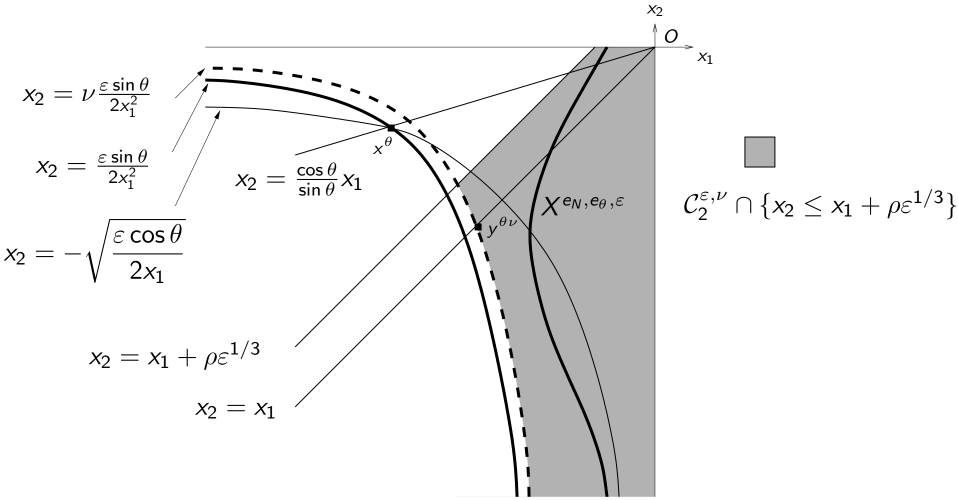

x_2
O
x_1
$x_2 = \nu \frac{\varepsilon \sin\theta}{2x_1^2}$
$x_2 = \frac{\varepsilon \sin\theta}{2x_1^2}$
$x_2 = -\sqrt{\frac{\varepsilon \cos\theta}{2x_1}}$
x^θ
$x_2 = \frac{\cos\theta}{\sin\theta} x_1$
$X^{e_N, e_\theta, \varepsilon}$
$y^{\theta\nu}$
$\mathcal{C}_2^{\varepsilon,\nu} \cap \{x_2 \le x_1 + \rho\varepsilon^{1/3}\}$
$x_2 = x_1 + \rho\varepsilon^{1/3}$
$x_2 = x_1$

FIGURE –

Failure of the semigroup property

The limit trajectories $X^{x,\alpha}$ do not satisfy the semigroup property and are not stable.

Corollary

Take $x \in \Gamma$ and $\alpha \in \mathcal{A}_x$.

1. (Failure of the semigroup property) *We may have*

$$X^{x,\alpha} \in \omega(x,\alpha)$$

$$X^{X^{x,\alpha}(t),\alpha(t+\cdot)} \in \omega(X^{x,\alpha}(t),\alpha(t+\cdot)) \quad \textit{for some} \quad t>0$$

and

$$X^{x,\alpha}(t+s) \neq X^{X^{x,\alpha}(t),\alpha(t+\cdot)}(s)$$

2. (Instability) *It can happen that $x_n \to x$ and $X^{x_n,\alpha}$ converges locally uniformly in $(0,\infty)$ to some $Y \neq X^{x,\alpha}$.*

- Take $\alpha(t) \equiv e_{5\pi/4}$ and $x = e_N$. Then, $X^{x,\alpha}(t)$ is explicit and, in particular, for $s \geq 0$,

$$X^{x,\alpha}(\sqrt{2}+s) = (-1-2^{-1/2}s, 0) \neq O \equiv X^{O,\alpha}(s) = X^{X^{x,\alpha}(\sqrt{2}),\alpha(\sqrt{2}+\cdot)}(s)$$

- Replace the starting point $x = e_N$ with $x_n = \frac{1}{n}e_N \to O$ as $n \to \infty$.

This provides a sequence of trajectories $X^{x_n,\alpha}$ still going to the branch W whereas $X^{O,\alpha} \equiv O$ is stuck at O.

Perspectives

- Uniqueness of the limit $(X^{x,\alpha}, k^{x,\alpha})$

- The proof of $V^\varepsilon \to \overline{V}$ relies on some optimal control techniques. Is it possible to obtain the result by PDE techniques, passing to the limit directly in the viscosity inequalities satisfied by V^ε ?

- Weaken the additional assumptions

$$\overline{B}(0,\eta) \subset f(x,A) \quad \text{for some} \quad \eta > 0$$

and

$$\ell(x,a) = \ell(x), \quad \forall\,(x,a) \in \mathbb{R}^2 \times A$$

- More general networks

Thank you for your attention !