Solutions to the Hamilton-Jacobi equation for dynamic optimization problems with discontinuous time dependence

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Outline of the talk

- Dynamic Optimization problems with discontinuous time dependence and Hamilton-Jacobi equation
- A characterization of the value function and Hamilton-Jacobi equation
- Optimal Control
- Calculus of Variations

joint work with J. Bernis, C. Mariconda, R. Vinter

Optimal control problems - Value function

Consider the optimal control problem:

$$(P_{S,x_0})\begin{cases} \text{Minimize } \int_S^T L(t,x(t),\dot{x}(t))\,dt + g(x(T)) \\ \text{over arcs } x(.) \in W^{1,1}([S,T];\mathbb{R}^n) \text{ satisfying} \\ \dot{x}(t) \in F(t,x(t)) \quad \text{a.e. } t \in [S,T] \\ x(S) = x_0, \end{cases}$$

Embed in family of problems, parameterized by initial data

$$(P_{t,x})\begin{cases} \text{Minimize } \int_t^T L(t,x(t),\dot{x}(t))\,dt + g(x(T)) \\ \text{over arcs } x(.) \text{ s.t. } \dot{x}(s) \in F(s,x(s)), \quad x(t) = x \end{cases}$$

Define $\boxed{V(t,x) = \text{Inf}(P_{t,x})}$ **Value Function**

Hamilton Jacobi equation

$$V(t,x) = \text{Inf}(P_{t,x}) \begin{cases} \text{Minimize } \int_t^T L(t,x(t),\dot{x}(t))\,dt + g(x(T)) \\ \text{over arcs } x(.) \text{ s.t. } \dot{x}(s) \in F(s,x(s)), \quad x(t) = x \end{cases}$$

A classical issue

- Characterize the value function as a solution (possibly unique) in a generalized sense of the HJE equation associated with $(P_{t,x})$

PDE of Dynamic Programming: $V(.,.)$ is a solution to

$$(HJE) \begin{cases} \nabla_t V(t,x) + \min_{v \in F(t,x)} [\nabla_x V(t,x) \cdot v + L(t,x,v)] \; = \; 0 \\ \qquad\qquad\qquad\qquad \forall (t,x) \in (S,T) \times \mathbb{R}^n \\ V(T,x) \; = \; g(x) \quad \forall x \in \mathbb{R}^n\,. \end{cases}$$

Characterize the value function as solution to HJE, in a generalized sense. **Two different classical paths:**

- **viscosity solutions approach:** (Crandall-Lions, Evans, Barles, ... ANR COSS members,...)
  - show that the value function is a viscosity solution (Frêchet sub/super-gradients, test-functions)
  - prove directly (without consideration of state trajectories) that the relevant HJE equation has a unique viscosity solution (comparison results)
- **system theoretic approach:** (Clarke, Frankowska, Vinter, ... ANR COSS members,...)
  - intimately connected with (monotonicity) properties of state trajectories
  - invariance (viability) theorems are employed to show that an arbitrary generalized ('proximal', 'Dini') solution to the HJE simultaneously majorizes and minorizes the value function and, therefore, coincides with it (**Nonsmooth Analysis**)

System theoretic approach - characterize lsc value functions

Theorem. [Frankowska, SICON 1993] F is required to be **continuous** w.r.t. time ($L = 0$). Then, V is the unique lsc function satisfying the HJE, in the sense ('**Dini solution**'):

(i): $$\inf_{v \in F(t,x)} D_{\uparrow} V((t,x);(1,v)) \leq 0,$$
$$\text{for all } (t,x) \in ([S,T) \times \mathbb{R}^n) \cap \operatorname{dom} V$$

(ii): $$\sup_{v \in F(t,x)} D_{\uparrow} V((t,x);(-1,-v)) \leq 0,$$
$$\text{for all } (t,x) \in ((S,T] \times \mathbb{R}^n) \cap \operatorname{dom} V$$

(iii): $V(T,x) = g(x)$ for all $x \in \mathbb{R}^n$.

$D_{\uparrow} V(.,.) \rightarrow$ **the lower Dini directional derivative** (also called contingent epi-derivative).

- Equivalent conditions involving generalized solutions to HJE in a **Frêchet subgradient** sense were also given in [Frankowska, SICON 1993]

Refined version in terms of '**proximal subgradients**'

Theorem. [Clarke-Ledyaev-Stern-Wolenski, J. Dynam. Control Systems 1995] F is required to be **continuous** w.r.t. time ($L = 0$). Then, V is the unique lsc function satisfying the HJE, in the sense ('**proximal solution**'):

(i): for all $(t, x) \in ((S, T) \times \mathbb{R}^n) \cap \mathrm{dom}\, V$, $(\xi^0, \xi^1) \in \partial_P V(t, x)$

$$\xi^0 + \inf_{v \in F(t,x)} \xi^1 \cdot v \;=\; 0,$$

(ii): for all $x \in \mathbb{R}^n$,

$$\liminf_{\{(t',x') \to (S,x)\,:\,t' > S\}} V(t', x') = V(S, x)$$

and

$$\liminf_{\{(t',x') \to (T,x)\,:\,t' < T\}} V(t', x') = V(T, x) = g(x).$$

A reason for lsc value functions

$$V(t,x) = \text{Inf}(P_{t,x}) \begin{cases} \text{Minimize } \int_t^T L(t,x(t),\dot{x}(t))\,dt + g(x(T)) \\ \text{over arcs } x(.) \text{ s.t. } \dot{x}(s) \in F(s,x(s)) \;\; x(t) = x \end{cases}$$

$\rightarrow g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is **extended valued**; incorporates an implicit **terminal constraint**

$$x(T) \in C\,,$$

where $C := \{x \in \mathbb{R}^n \;:\; g(x) < +\infty\}$ is a closed set.

$\Rightarrow$ It is necessary to consider lower semicontinuous solutions (lsc) to HJE

Discontinuous time-dependent problems

Generalized solution to HJE in an 'almost everywhere' sense?

Example. Consider ($L = 0$)

$$\begin{cases} \text{Minimize } g(x(1)) := x(1) \\ \text{over arcs } x(.) \in W^{1,1}([0,1];\mathbb{R}) \text{ s.t.} \\ \dot{x}(t) = 0 \quad \text{a.e. } t \in [0,1] \\ x(0) = x_0 \,, \end{cases}$$

The **value function** is $V(t,x) = x$ for all (t,x).

However

$$W(t,x) := \begin{cases} x - 1 & \text{if } t \leq \frac{1}{2} \\ x & \text{if } t > \frac{1}{2} \end{cases}$$

is also an lsc function that is a '**Dini solution**' in the 'almost everywhere' sense: we exclude consideration of the troublesome point $\frac{1}{2}$ at which $W(t,x)$ fails to satisfy conditions (i) and (ii) above.

$\Rightarrow$ the **value function is not the unique lsc Dini/proximal solution in the almost everywhere sense.**

The **non-uniqueness issue** can be circumvented by restricting candidate solutions $V(.,.)$ to have the following regularity property ([Frankowska, Plaskacz and Rzezuchowski, JDE 1995]):

(EPI) $t \to \text{epi}\, V(t,.)$ is absolutely continuous.

$\to$ $\text{epi}\, V(t,.) := \{(\alpha, x) \,:\, \alpha \geq V(t,x)\}$ and 'absolute continuity' means that there exists an integrable function $\gamma(.) : [S,T] \to \mathbb{R}$ such that

$$d_H(\text{epi}\, V(s,.), \text{epi}\, V(t,.)) \leq \int_{[s,t]} \gamma(\sigma) d\sigma\,, \quad \text{for all } [s,t] \subset [S,T]\,.$$

($d_H(.,.)$ denotes the Hausdorff distance.)

$\to$ [Vinter-Wolenski, SICON 1990]: HJ theory for optimal control problems with data measurable in time ('verification Thm.')

The 'almost everywhere' HJE theory of Frankowska et al. covers a broad class of optimal control problems for which $t \to F(t,x)$ is discontinuous.

BUT it leaved open the following **question**:

> *For the special case, when $t \to F(t,x)$ **and** $t \to L(t,x,v)$ **have everywhere one-sided limits and is continuous on the complement of a zero-measure subset of** $[S,T]$, can we provide a characterization of the value function as a unique lsc* **'generalized solution', without imposing the a priori regularity condition (EPI) on** $V(.,.)$*?*

Positive answers: [P.B.-Vinter, SICON 2017], [Bernis-P.B., ESAIM COCV 2020], [Bernis-P.B.-Vinter, JDE 2022], [Bernis-P.B. JCA 2023], [P.B.-Vinter, Springer SMM 2024]

Our framework - Enter also a state constraint

$$(P)\begin{cases} \text{Minimize } \int_t^T L(t,x(t),\dot{x}(t))\,dt + g(x(T)) \\ \text{over arcs } x \in W^{1,1}([S,T];\mathbb{R}^n) \text{ satisfying} \\ \dot{x}(t) \in F(t,x(t)) \quad \text{a.e. } t \in [S,T] \\ x(t) \in A \quad \textbf{for all } t \in [S,T] \qquad \leftarrow \quad \textbf{state constraint} \\ x(S) = x_0\,. \end{cases}$$

$\rightarrow$ **state constraint**: A is a nonempty closed set in $\mathbb{R}^n$

$\rightarrow$ **the Lagrangian L is merely continuous w.r.t. x**

Example: A Growth/Consumption Model

A 'growth versus consumption' problem of neoclassical macro-economics, based on the Ramsey model of economic growth.

Question: what balance should be struck between investment and consumption to **maximize overall investment in social programmes** over a fixed period of time?

$$\begin{cases} \textbf{Maximize} \ \int_0^T (1-u(t))x^\alpha(t)dt \\ \text{subject to} \\ \dot{x}(t) = -ax(t) + bu(t)x^\alpha(t) \quad \text{for a.e. } t \in [0,T], \\ u(t) \in [0,1] \quad \text{for a.e. } t. \in [0,T]\,, \\ x(t) \geq 0 \text{ for all } t \in [0,T]\,, \\ x(0) = x_0\ . \end{cases}$$

Here, $a > 0$, $b > 0$, $x_0 \geq 0$ and $\alpha \in (0,1)$ are given constants and $[0,T]$ is a given interval.

A Growth/Consumption Model...

$$\begin{cases} \textbf{Maximize } \int_0^T (1-u(t))x^\alpha(t)dt \\ \text{subject to} \\ \dot{x}(t) = -ax(t) + bu(t)x^\alpha(t) \quad \text{for a.e. } t \in [0,T], \\ u(t) \in [0,1] \quad \text{for a.e. } t. \in [0,T]\,, \\ x(t) \geq 0 \text{ for all } t \in [0,T]\,, \\ x(0) = x_0\ . \end{cases}$$

Data/model interpretation:

$x \rightarrow$ **global economic output**
$r(x) = bx^\alpha \rightarrow$ financial return from economic output x
$-ax \rightarrow$ fixed costs reducing growth
$u \rightarrow$ **the proportion to invest in industry**
$1-u \rightarrow$ **the proportion to invest in social programmes**

A Growth/Consumption Model...

$$\begin{cases} \textbf{Minimize} \ -\int_0^T L(y(t),\dot{y}(t))\,dt \\ \text{subject to} \\ \dot{y}(t) \in F(y(t)) \quad \text{for a.e. } t \in [0,T], \\ -y(t) \leq 0 \text{ for all } t \in [0,T], \leftarrow \quad \textbf{state constraint} \\ y(0) = y_0 \ . \end{cases}$$

$\rightarrow$ **transformation** $y = x^{1-\alpha}$

$F(y) := \{v \,:\, v = (1-\alpha)(-ay + bu) \text{ for some } u \in [0,1]\}$ and

$$L(y,v) := \Big(1 - b^{-1}((1-\alpha)^{-1}v + ay)\Big)\, y^{\frac{\alpha}{1-\alpha}} \ .$$

Rmk: L is NOT Lipschitz in y (in general), only continuous...

Our Hypotheses

(H1): $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is lsc, $F : [S, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ takes closed, convex, non-empty values, $F(., x)$ is $\mathcal{L}$-measurable for all $x \in \mathbb{R}^n$, $L : [S, T] \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is $\mathcal{L} \times \mathcal{B}^{n+n}$-measurable and $L(t, x, .)$ is convex for every $t \in [S, T]$ and $x \in \mathbb{R}^n$,

(H2): **(i)** there exists $c_F \in L^1(S, T)$ such that

$$F(t, x) \subset c_F(t)(1 + |x|)\, \mathbb{B}$$

for all $x \in \mathbb{R}^n$ and for a.e. $t \in [S, T]$,

and

(ii) for every $R_0 > 0$, there exists $c_0 > 0$ such that

$$F(t, x) \subset c_0\, \mathbb{B} \quad \text{for all } (t, x) \in [S, T] \times R_0\mathbb{B} \,,$$

Hypotheses...

(H3): **(i)** for every $R_0 > 0$, there exists a modulus of continuity $\omega(.) : \mathbb{R}^+ \to \mathbb{R}^+$ and $k_F \in L^1(S,T)$ such that

$$d_H(F(t,x'), F(t,x)) \leq \omega(|x-x'|) \quad \text{for all } x, x' \in R_0\mathbb{B} \ ,$$

and

(ii)
$$F(t,x') \subset F(t,x) + k_F(t)|x-x'|\,\mathbb{B}$$
$$\text{for all } x, x' \in R_0\mathbb{B} \text{ and a.e. } t \in [S,T]\,,$$

(H4): **(i)** for each $s \in [S,T)$, $t \in (S,T]$ and $x \in \mathbb{R}^n$ the following one-sided set-valued limits exist and are non-empty:

$$F(s^+,x) := \lim_{s' \downarrow s} F(s',x), \quad F(t^-,x) := \lim_{t' \uparrow t} F(t',x) \ ,$$

and

(ii) and for a.e. $s \in [S,T)$ and $t \in (S,T]$ we have

$$F(s^+,x) = F(s,x), \quad F(t^-,x) = F(t,x) \ , \quad \text{for all } x \in \mathbb{R}^n \ .$$

Hypotheses...

(H5): **(i)** for each $s \in [S,T)$, $t \in (S,T]$, $x \in \mathbb{R}^n$, $v \in \mathbb{R}^n$ the following limits exist

$$L(s^+,x,v), \quad L(t^-,x,v)\ ,$$

(ii) for a.e. $s \in [S,T)$ and $t \in (S,T]$ we have

$$L(s^+,x,v) = L(s,x,v), \quad L(t^-,x,v) = L(t,x,v)\ ,$$

(iii) there exist $c_L \geq c_0$ (c_0 is the constant of (H2)(ii)), $M_L > 0$ and a modulus of continuity ω_L such that

$$|L(t,x,v)| \leq M_L, \quad \text{for all} \quad (t,x,v) \in [S,T] \times R_0\mathbb{B} \times 2c_L\mathbb{B}$$

and

$|L(t,x',v) - L(t,x,v)| \leq \omega_L(|x-x'|)$, for all $x, x' \in R_0\mathbb{B}$, $t \in [S,T]$ and $v \in c_L\mathbb{B}$.

F of bounded variation

$F(\cdot, x)$ is of **bounded variation (BV)** uniformly over $x \in R_0\mathbb{B}$, if

$$d_H(F(s,x), F(t,x)) \leq \eta(t) - \eta(s), \;\; \forall [s,t] \subset [S,T], \;\; x \in R_0\mathbb{B}$$

for some non-decreasing function of bounded variation $\eta : [S,T] \to [0,\infty)$

Remark If $F(\cdot, x)$ is of **bounded variation (BV)** uniformly over $x \in R_0\mathbb{B}$ then $t \to F(t,x)$ has everywhere one-sided limits and is continuous on the complement of a zero-measure subset of $[S,T]$, i.e. **(H4) is satisfied**

Constraint qualifications

$(CQ)_{BW}$: There exists a modulus of continuity $\tilde{\theta}(\cdot)$, such that given any interval $[t_0, t_1] \subset [S, T]$, any F-trajectory $\hat{x}(\cdot)$ on $[t_0, t_1]$ with $\hat{x}(t_1) \in A$, and any $\rho > 0$ such that $\rho \ \geq \ \max\{d_A(\hat{x}(t)) \ : \ t \in [t_0, t_1]\}$, we can find an F-trajectory $x(\cdot)$ on $[t_0, t_1]$ such that $x(t_1) = \hat{x}(t_1)$, $x(t) \ \in \ \text{int}\, A \quad$ for all $t \in [t_0, t_1)$ and

$$\|\hat{x} - x\|_{W^{1,1}(t_0,t_1)} \leq \tilde{\theta}(\rho).$$

$(CQ)_{FW}$: There exists a modulus of continuity $\theta(\cdot)$, such that given any interval $[t_0, t_1] \subset [S, T]$, any F-trajectory $\hat{x}(\cdot)$ on $[t_0, t_1]$ with $\hat{x}(t_0) \in A$, and any $\rho > 0$ such that $\rho \ \geq \ \max\{d_A(\hat{x}(t)) \ : \ t \in [t_0, t_1]\}$, we can find an F-trajectory $x(\cdot)$ on $[t_0, t_1]$ such that $x(t_0) = \hat{x}(t_0)$, $x(t) \ \in \ \text{int}\, A \quad$ for all $t \in (t_0, t_1]$ and

$$\|\hat{x} - x\|_{W^{1,1}(t_0,t_1)} \leq \theta(\rho).$$

Geometric Conditions for $(CQ)_{BW}$ and $(CQ)_{FW}$

The 'outward' and 'inward' pointing conditions ((OPC) and (IPC)) + 'additional hypotheses on the data' yield 'interiority' hypotheses: either 'backward' $(CQ)_{BW}$ or 'forward'$(CQ)_{FW}$

(OPC): for each $s \in [S, T)$, $t \in (S, T]$ and $x \in \partial A$,

$$F(t^-, x) \cap (-\mathrm{int}\, T_A(x)) \neq \emptyset \,, \qquad F(s^+, x) \cap (-\mathrm{int}\, T_A(x)) \neq \emptyset \,;$$

(IPC): for each $s \in [S, T)$, $t \in (S, T]$ and $x \in \partial A$,

$$F(t^-, x) \cap \mathrm{int}\, T_A(x) \neq \emptyset \,, \qquad F(s^+, x) \cap \mathrm{int}\, T_A(x) \neq \emptyset \,.$$

The **Clarke tangent cone** to A at x:

$$T_A(x) := \liminf_{t \downarrow 0, y \xrightarrow{A} x} t^{-1}(A - y).$$

Theorem 1 [Bernis-P.B.-Vinter, JDE 2022]
Assume (H1)–(H5) and $(CQ)_{BW}$. Take a function $V : [S,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. Then, assertions (a)-(b)-(c) below are equivalent:

(a) V is the value function for (P).

(b) V is lsc on $[S,T] \times \mathbb{R}^n$ and

(i) for all $(t,x) \in ([S,T) \times A) \cap \mathrm{dom}\, V$

$$\inf_{v \in F(t^+,x)} \Big[D_{\uparrow} V((t,x);(1,v)) + L(t^+,x,v) \Big] \;\le\; 0,$$

(ii) for all $(t,x) \in ((S,T] \times \mathrm{int}\, A) \cap \mathrm{dom}\, V$

$$\sup_{v \in F(t^-,x)} \Big[D_{\uparrow} V((t,x);(-1,-v)) - L(t^-,x,v) \Big] \;\le\; 0,$$

(iii) for all $x \in A$

$$\liminf_{\{(t',x') \to (T,x)\,:\, t'<T, x' \in \mathrm{int}\, A\}} V(t',x') = V(T,x) = g(x).$$

Characterization of lsc Value Functions...

(c) V is lsc on $[S,T]\times\mathbb{R}^n$ and

(i) for all $(t,x)\in((S,T)\times A)\cap\mathrm{dom}\,V$, $(\xi^0,\xi^1)\in\partial_P V(t,x)$

$$\xi^0+\min_{v\in F(t^+,x)}\left[\xi^1\cdot v+L(t^+,x,v)\right]\ \le\ 0,$$

(ii) for all $(t,x)\in((S,T)\times\mathrm{int}\,A)\cap\mathrm{dom}\,V$, $(\xi^0,\xi^1)\in\partial_P V(t,x)$

$$\xi^0+\min_{v\in F(t^-,x)}\left[\xi^1\cdot v+L(t^-,x,v)\right]\ge 0,$$

(iii) for all $x\in A$,

$$\liminf_{\{(t',x')\to(S,x):t'>S\}} V(t',x')=V(S,x)$$

and

$$\liminf_{\{(t',x')\to(T,x)\ :\ t'<T,x'\in\mathrm{int}\,A\}} V(t',x')=V(T,x)=g(x).$$

Characterization of the Value Function

Theorem 2 [Bernis-P.B.-Vinter, JDE 2022]
Assume (H1)–(H5) and and $(CQ)_{FW}$ are satisfied. Assume, furthermore that g **is continuous on** A. Let $V : [S, T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended valued function. Then the assertions (a), (b) and (c) of Theorem 1 remain equivalent.

Questions:

- Exchange the limits of F and L?
- Suppose that (IPC)+'additional hypotheses' $\Rightarrow$ $(CQ)_{FW}$ is in force. Can we remove g **is continuous on** A?

Exchange the limits of F?

Example 1. Consider the optimal control problem ($L = 0$, $A = \mathbb{R}$)

$$\begin{cases} \text{Minimize } g(x(1)) := x(1) \\ \text{over arcs } x(.) \in W^{1,1}([t_0, 1]; \mathbb{R}) \text{ s.t.} \\ \dot{x}(t) \ \in \ F(t) \quad \text{a.e. } t \in [t_0, 1] \\ x(t_0) \ = \ x_0 \ , \end{cases}$$

where $t_0 \in [0, 1]$, $x_0 \in \mathbb{R}$ and

$$F(t) \ := \begin{cases} [-\frac{1}{2}, \frac{1}{2}] & \text{if} \quad 0 \leq t \leq \frac{1}{2} \\ [-1, 1] & \text{if} \quad \frac{1}{2} < t \leq 1 \ . \end{cases}$$

The value function $V : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is

$$V(t, x) \ := \begin{cases} x + \frac{t}{2} - \frac{3}{4} & \text{if} \quad 0 \leq t \leq \frac{1}{2} \\ x + t - 1 & \text{if} \quad \frac{1}{2} < t \leq 1 \ . \end{cases}$$

We have, as the result of a routine calculation:

$$D_{\uparrow} V((1/2, 0); (1, v)) \ = \ 1 + v, \quad D_{\uparrow} V((1/2, 0); (-1, -v)) \ = \ -\frac{1}{2} - v \ .$$

Exchange the limits of F?...

Consistent with conditions (b)(i) and (b)(ii) in Thm. above, V satisfies

$$\inf_{v\in F(\frac{1}{2}^+)} D_\uparrow V((1/2,0);(1,v)) \;=\; \inf_{v\in[-1,1]}(1+v) \;=\; 0\,,$$

$$\sup_{v\in F(\frac{1}{2}^-)} D_\uparrow V((1/2,0);(-1,-v)) \;=\; \sup_{v\in[-\frac{1}{2},\frac{1}{2}]} (-\frac{1}{2}-v) = \;0\,.$$

On the other hand, switching roles of $F(\frac{1}{2}^-)$ and $F(\frac{1}{2}^+)$ in these calculations would give:

$$\inf_{v\in F(\frac{1}{2}^-)} D_\uparrow V((1/2,0);(1,v)) \;=\; \inf_{v\in[-\frac{1}{2},\frac{1}{2}]}(1+v) \;=\; \frac{1}{2}\;\;(>0)\,,$$

$$\sup_{v\in F(\frac{1}{2}^+)} D_\uparrow V((1/2,0);(-1,-v)) \;=\; \sup_{v\in[-1,1]} (-\frac{1}{2}-v) \;=\; \frac{1}{2}\;\;(>0)\,.$$

$\Rightarrow$ condition **(b)(i) must involve the right limit** $F(t^+,x)$ and **(b)(ii) must involve the left limit** $F(t^-,x)$.

'g is continuous' hypothesis cannot be removed

Example 2. Let $n = 1$, $[S, T] = [0, 1]$, $A = \{x \in \mathbb{R}, x \geq 0\}$, $F(t, x) \equiv [0, 1]$, $L = 0$ and

$$g(x) = \begin{cases} -x - 2, & \text{if } x \leq 0 \\ -x, & \text{if } x > 0\,. \end{cases}$$

Observe that (H1)-(H5), (IPC) are each one of the supplementary hypotheses are satisfied. But the hypothesis 'g is continuous' is violated. The value function is

$$V(t, x) = \begin{cases} t - x - 1, & \text{if} \quad x > 0 \\ -2, & \text{if} \quad x = 0 \\ +\infty, & \text{if} \quad x < 0 \end{cases}, \qquad \text{for all } (t, x) \in [0, 1] \times \mathbb{R}\,.$$

Notice that $\liminf_{\{(t',x')\to(1,0)\,|\,x'>0\}} V(t', x') = 0 \neq V(1, 0) = -2$. Therefore, the value function does not satisfy condition (iii) of (b) and (c).

Viscosity Solution - 'Forward (CQ)'

Theorem 3 [Bernis-P.B., JCA 2023]

Assume (H1)–(H5) and and $(CQ)_{FW}$ are satisfied. Assume, furthermore that g **is continuous on** A. Then the value function V is characterized by

(d) V is continuous on $[S,T]\times A$, satisfies $V(t,x)=+\infty$ whenever $x\notin A$ and

(i) for all $(t,x)\in(S,T)\times A$, $(\xi^0,\xi^1)\in\partial_- V(t,x)$

$$\xi^0+\inf_{v\in F(t^+,x)}\left[\xi^1\cdot v+L(t^+,x,v)\right]\le 0\,;$$

(ii) for all $(t,x)\in(S,T)\times \mathrm{int}\,A$, $(\xi^0,\xi^1)\in\partial_+ V(t,x)$

$$\xi^0+\inf_{v\in F(t^+,x)}\left[\xi^1\cdot v+L(t^+,x,v)\right]\ge 0\,;$$

(iii) for all $x\in A$

$$\liminf_{\{(t',x')\to(S,x)\,:\,t'>S\}} V(t',x')=V(S,x),$$

$$\text{and}\qquad V(T,x)=g(x).$$

Theorem 4 [Bernis-P.B., JCA 2023]

Assume (H1)–(H5), $(CQ)_{BW}$ **and** $(CQ)_{FW}$. Suppose, in addition, that $g_{|A}$ **is locally bounded and satisfies** $((g_{|A})^*)_* = g_{|A}$. Then the value function V is characterized by

(d)′ V is lsc on $[S,T]\times\mathbb{R}^n$ and locally bounded on $[S,T]\times A$, satisfies $V(t,x)=+\infty$ whenever $x\notin A$ and

(i) for all $(t,x)\in(S,T)\times A$, $(\xi^0,\xi^1)\in\partial_- V(t,x)$

$$\xi^0+\inf_{v\in F(t^+,x)}\left[\xi^1\cdot v+L(t^+,x,v)\right]\le 0\,; \qquad (1)$$

(ii) for all $(t,x)\in(S,T)\times\operatorname{int}A$, $(\xi^0,\xi^1)\in\partial_+ V^*(t,x)$

$$\xi^0+\inf_{v\in F(t^+,x)}\left[\xi^1\cdot v+L(t^+,x,v)\right]\ge 0\,; \qquad (2)$$

(iii) for all $x\in A$

$$\liminf_{\{(t',x')\to(S,x)\,:\,t'>S\}} V(t',x')=V(S,x),$$

$$(V_{|[S,T]\times A})^*(T,x)=(g_{|A})^*(x) \quad \text{and} \quad V(T,x)=g(x).$$

Notation

If $f : \mathbb{R}^m \to \mathbb{R}$ is a locally bounded function, we denote its **lower (resp. upper) semicontinous envelope** by:

$$f_*(x) := \liminf_{y \to x} f(y) \quad \left(\text{resp. } f^*(x) := \limsup_{y \to x} f(y) \right).$$

Take a lsc function $\varphi : \mathbb{R}^k \to \mathbb{R} \cup \{+\infty\}$ and points $x \in \operatorname{dom} \varphi$. The **Fréchet (strict) subdifferential**:

$$\partial_- \varphi(x) := \left\{ \xi \in \mathbb{R}^k \ : \ \limsup_{y \to x} \frac{\xi \cdot (y - x) - (\varphi(y) - \varphi(x))}{|y - x|} \leq 0. \right\}$$

If $\varphi : \mathbb{R}^k \to \mathbb{R} \cup \{-\infty\}$ is an upper semicontinuous function and $x \in \operatorname{dom} \varphi$, then the **Fréchet superdifferential** of φ at x is

$$\partial_+ \varphi(x) := -\partial_-(-\varphi)(x)$$

Write $V : [0,T] \times (0,\infty) \to \mathbb{R}$ the value function for (GC).
Let $\psi : [0,\infty) \to [0,\infty)$ be the mapping

$$\psi(x) := x^{1-\alpha} \text{ for } x \in [0,\infty)\,.$$

Then

$$V(t,x) = (W \circ (Id,\psi))(t,x), \quad \text{for all } (t,x) \in [0,T] \times [0,\infty)\,,$$

where $W : [0,T] \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is the unique upper semicontinuous function s.t. $W(t,y) = -\infty$ whenever $y < 0$,

(i) for all $(t,y) \in (0,T) \times [0,\infty)$, $(\xi^0,\xi^1) \in \partial^P W(t,y)$

$$\xi^0 + \sup_{u\in[0,1]} \left(\xi^1 \cdot (-a(1-\alpha)y + (1-\alpha)bu) + (1-u)y^{\frac{\alpha}{1-\alpha}} \right) \geq 0;$$

(ii) for all $(t,y) \in (0,T) \times (0,\infty)$, $(\xi^0,\xi^1) \in \partial^P W(t,y)$

$$\xi^0 + \sup_{u\in[0,1]} \left(\xi^1 \cdot (-a(1-\alpha)y + (1-\alpha)bu) + (1-u)y^{\frac{\alpha}{1-\alpha}} \right) \leq 0;$$

$\partial^P W(t,y) = -\partial_P(-W)(t,y)$: **proximal superdifferential** of W

Proximal solution...

(iii) for all $y \in [0, \infty)$

$$\limsup_{\{(t',y')\to(0,y):t'>0\}} W(t', y') = W(0, y)$$

and

$$\limsup_{\{(t',y')\to(T,x):t'<T,\ y'>0\}} W(t', y') = W(T, y) = 0.$$

Growth/Consumption Example - Viscosity solution

$W : [0,T] \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is the unique upper semicontinuous function such that W is continuous on $[0,T] \times [0,\infty)$, $W(t,y) = -\infty$ whenever $y < 0$ and

(i) for all $(t,y) \in (0,T) \times [0,\infty)$, $(\xi^0, \xi^1) \in \partial_+ W(t,y)$

$$\xi^0 + \sup_{u \in [0,1]} \left(\xi^1 \cdot (-a(1-\alpha)y + (1-\alpha)bu) + (1-u)y^{\frac{\alpha}{1-\alpha}} \right) \geq 0;$$

(ii) for all $(t,y) \in (0,T) \times (0,\infty)$, $(\xi^0, \xi^1) \in \partial_- W(t,y)$

$$\xi^0 + \sup_{u \in [0,1]} \left(\xi^1 \cdot (-a(1-\alpha)y + (1-\alpha)bu) + (1-u)y^{\frac{\alpha}{1-\alpha}} \right) \leq 0;$$

(iii) for all $y \in [0,\infty)$

$$\limsup_{\{(t',y') \to (0,y),\, t'>0\}} W(t',y') = W(0,y)$$

and

$$W(T,y) = 0.$$

Calculus of Variations problems - Value function

Consider the Calculus of Variations problem:

$$(CV_{S,x_0})\begin{cases} \text{Minimize } \int_S^T L(t,x(t),\dot{x}(t))\,dt + g(x(T)) \\ \text{over arcs } x(.) \in W^{1,1}([S,T];\mathbb{R}^n) \text{ satisfying} \\ x(S)=x_0, \end{cases}$$

Embed in family of problems, parameterized by initial data

$$(CV_{t,x})\begin{cases} \text{Minimize } \int_t^T L(t,x(t),\dot{x}(t))\,dt + g(x(T)) \\ \text{over arcs } x(.) \in W^{1,1} \text{ s.t. } x(t)=x \end{cases}$$

Define $\boxed{V(t,x)=\text{Inf}(CV_{t,x})}$ **Value Function**

Hamilton Jacobi equation

$$V(t,x) = \text{Inf}(CV_{t,x}) \begin{cases} \text{Minimize } \int_t^T L(t,x(t),\dot{x}(t))\,dt + g(x(T)) \\ \text{over arcs } x(.) \in W^{1,1} \text{ s.t. } x(t) = x \end{cases}$$

Classical issues

- Establish regularity properties of the value function
- Characterize the value function as a solution in a generalized (possibly unique) sense of the HJE equation associated with $(CV_{t,x})$

PDE of Dynamic Programming: $V(.,.)$ is a solution to

$$(HJE) \begin{cases} \nabla_t V(t,x) + \inf_{v \in \mathbb{R}^n} [\nabla_x V(t,x) \cdot v + L(t,x,v)] = 0 \\ \qquad\qquad\qquad\qquad \forall (t,x) \in (S,T) \times \mathbb{R}^n \\ V(T,x) = g(x) \quad \forall x \in \mathbb{R}^n \,. \end{cases}$$

Some results in the Calculus of Variations

- **Galbraith, SICON 2000** (Regularity assumptions directly on the Hamiltonian)
- **Dal Maso-Frankowska, ESAIM 2000, AMO 2003**
  - Autonomous case, $L = L(x, v)$ is Borel, locally bounded, superlinear and convex in v, g is lsc, $(CV_{t,x})$ has a minimizer for all (t, x)
  - the value function is lsc and a 'Dini' and a viscosity solution + a 'partial comparison' result (the value function is the greatest lcs Dini subsolution)
- **Plaskacz-Quincampoix, Top. Math. Non. An., 2002**
  - Nonautonomous case, L is continuous, locally bounded, superlinear and convex in v, Lipschitz regularity in x, $(CV_{t,x})$ has a Lipschitz minimizer for all (t, x)
  - the value function is lsc and a 'Dini', proximal and a viscosity solution + uniqueness if V is bounded from below

Our Setting

- $L : [S,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is Lebesgue/Borel
- L is locally bounded and has linear growth from below: $\exists \alpha, d > 0$ s.t.
$$L(s,y,v) \geq \alpha |v| - d$$
- Growth condition: for all $K \geq 0$,
$$\lim_{\substack{|v| \to +\infty \\ P(s,y,v) \in \partial_\mu \left(L\left(s,y,\frac{v}{\mu}\right)\mu \right)_{\mu=1} \neq \emptyset}} P(s,y,v) = -\infty \ \text{ unif. } |y| \leq K,$$
meaning that for all $M \in \mathbb{R}$ there exists $R > 0$ such that $P(s,y,v) \leq M$ for all $(s,y,v) \in [S,T] \times \mathbb{R}^n \times \mathbb{R}^n$ with $|y| \leq K$, $|v| \geq R$, $\partial_\mu (L(s,y,\frac{v}{\mu})\mu)_{\mu=1} \neq \emptyset$ and $P(s,y,v) \in \partial_\mu (L(s,y,\frac{v}{\mu})\mu)_{\mu=1}$.

- $(CV_{t,x})$ has a minimizer for all (t, x)
- L has a **bounded variation** behaviour: there exist $\kappa, A \geq 0, \gamma \in L^1([S,T])$, $\varepsilon_* > 0$ and a non-decreasing function $\eta : [S,T] \to [0,+\infty)$ satisfying, for a.e. $s \in [S,T]$

$$
\begin{aligned}
|L(s_2, y, v) - L(s_1, y, v)| \leq \eta(s_2) - \eta(s_1) \\
+ \big(\kappa L(s, y, v) + A|v| + \gamma(s)\big)\, |s_2 - s_1|
\end{aligned}
$$

whenever $s_1 \leq s_2$ belong to $[s - \varepsilon_*, s + \varepsilon_*] \cap [S,T]$, $y \in \mathbb{R}^n$, $v \in \mathbb{R}^n$.

Theorem 5 [Bernis-P.B.-Mariconda, 2024]

- V is lsc on $[S,T]\times\mathbb{R}^n$ and locally Lipschitz on $[S,T[\times\mathbb{R}^n$.
- V is a Dini and proximal solution
- If U is a Dini subsolution, then $U\leq V$.
- If, in addition, L is convex in v , V is bounded from below and is a Dini supersolution, then $V\leq U$. (This implication required a new invariance/viability theorem.)

(a) V is a Dini solution in the following sense:

(i) for any $(t,x) \in ([S,T) \times \mathbb{R}^n) \cap \mathrm{dom}(V)$, there exist $\epsilon_{t,x},\ R_{t,x} > 0$ such that, for all $(t',x') \in ((t,x) + \epsilon_{t,x}\mathbb{B}) \cap ([S,T) \times \mathbb{R}^n) \cap \mathrm{dom}(V)$, we can find $v' \in R_{t,x}\mathbb{B}$ satisfying:

$$\mathrm{D}_{\uparrow} V((t',x'),(1,v')) + L^{\flat}(t',x',v') \leq 0;$$

(ii) for any $(t,x) \in ((S,T] \times \mathbb{R}^n) \cap \mathrm{dom}(V)$

$$\sup_{v \in \mathbb{R}^n} [\mathrm{D}_{\downarrow} V((t,x),(-1,-v)) - L^{\sharp}(t,x,v)] \leq 0;$$

(iii) for all $x \in \mathbb{R}^n$, $V(T,x) = g(x)$.

(b) V is a proximal solution in the following sense:

(i) for every $(t,x) \in ((S,T) \times \mathbb{R}^n) \cap \mathrm{dom}(V)$, there exist $\epsilon_{t,x},\ R_{t,x} > 0$ such that, for all $(t',x') \in ((t,x) + \epsilon_{t,x}\mathbb{B}) \cap ([S,T) \times \mathbb{R}^n) \cap \mathrm{dom}(V)$, we can find $v' \in R_{t,x}\mathbb{B}$ satisfying:

$$\xi^0 + \xi^1 \cdot v' + L^\flat(t',x',v') \le 0, \text{ for all } (\xi^0,\xi^1) \in \partial_P V(t',x')\,;$$

(ii) for every $(t,x) \in ((S,T) \times \mathbb{R}^n) \cap \mathrm{dom}(V)$:

$$\xi^0 + \inf_{v \in \mathbb{R}^n} \left[\xi^1 \cdot v + L^\sharp(t,x,v)\right] \ge 0, \text{ for all } (\xi^0,\xi^1) \in \partial_P V(t,x)\,; \tag{3}$$

(iii) for every $x \in \mathbb{R}^n$,

$$\liminf_{\{(t',x')\to(S,x)\,:\,t'>S\}} V(t',x') = V(S,x),$$

and

$$\liminf_{\{(t',x')\to(T,x)\,:\,t'<T\}} V(t',x') = V(T,x) = g(x).$$

Auxiliary Lagrangians

For every $(t, x, v) \in (S, T] \times \mathbb{R}^n \times \mathbb{R}^n$ we define

$$L^\sharp(t,x,v) := \limsup_{h\downarrow 0} \frac{1}{h} \inf \Big\{ \int_{t-h}^{t} L(\tau, z(\tau), \dot z(\tau))\, d\tau \text{ s.t. } z \in W^{1,1}([t-h,t];\mathbb{R}^n), \begin{cases} z(t) = x, \\ z(t-h) = x - hv \end{cases} \Big\}.$$

Similarly, for every $(t, x, v) \in [S, T) \times \mathbb{R}^n \times \mathbb{R}^n$ we define:

$$L^\flat(t,x,v) := \liminf_{h\downarrow 0} \frac{1}{h} \inf \Big\{ \int_{t}^{t+h} L(\tau, z(\tau), \dot z(\tau))\, d\tau \text{ s.t. } z \in W^{1,1}([t,t+h];\mathbb{R}^n), \begin{cases} z(t) = x, \\ z(t+h) = x + hv \end{cases} \Big\}.$$

These are extensions to the nonautonomous case of the auxiliary Lagrangians defined in [Dal Maso-Frankowska, 2003].

Thank you for your attention!