Approximation to Mean Field Games

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March 22nd 2024

Journée ANR COSS

Motivation: Many agent system

In today's interconnected world, systems involving numerous agents are prevalent.

Visual examples:

Crowd motion

Traffic flow

Flocking

Distributed AI systems

Other examples:

Markets

Financial market

Energy production

Networks

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Markets

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Networks

Challenge: How to introduce an optimality notion to these systems

The MFG system

The mean field game system is given by

$$\begin{cases} -\partial_t u - \nu\Delta u + H(x, Du) = F(x, m(t)) & \text{in } (0,T)\times\mathbb{R}^d \\ \partial_t m - \nu\Delta m - \operatorname{div}(mD_pH(x,Du)) = 0 & \text{in } (0,T)\times\mathbb{R}^d \\ m(0,x) = m_0(x), \quad u(T,x) = G(x, m(T)) & \text{in } \mathbb{R}^d, \end{cases}$$

- The first equation is the Hamilton Jacobi Bellman equation for the agents′ value function u.
- The second equation is the Fokker-Planck equation for the distribution of agents. $m(t)$ is the probability density of the state of players at time t
- $m_0 \in \mathcal{P}(\mathbb{R}^d)$ can be seen as the initial distribution of the agents.

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- $m_0 \in \mathcal{P}(\mathbb{R}^d)$ can be seen as the initial distribution of the agents.
- The **MFG** equilibrium is (u, m) solution to the above system.
- **Forward-Backward system**.
- The linearized version of the HJB equation is the adjoint equation of the Fokker-Plank equation.

References

- MFGs were introduced in 2006 by J. M. Lasry and P. L. Lions. and by M. Huang, R. P. Malhamé, and P. E. Caines.

- Useful references:

  - J. M. Lasry and P. L. Lions. Jeux à champ moyen I. Le cas stationnaire. C. R. Math. Acad. Sci. Paris,2006.
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  - Numerical analysis of PDEs makes possible the approximation of equilibria of complex systems.

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  - Construction of approximation of Nash equilibria (in feedback form) for N-persons games through the solution of the MFG system.

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- Impact of the theory:
  - Numerical analysis of PDEs makes possible the approximation of equilibria of complex systems.
  - Construction of approximation of Nash equilibria (in feedback form) for N-persons games through the solution of the MFG system.
  - Applications: finance, market economics (oil producers, carbon markets...), engineering (smart grids...), crowd dynamics, socio-politics (learning, opinion formation etc...)

Our contribution

Goal of this talk: Discuss some numerical methods to solve MFG

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- ✓ Lagrange-Galerkin method to solve the first order MFG PDE system ($\nu = 0$).
- ✓ Newton iterations to solve the second order MFG PDE system ($\nu > 0$).

Lagrange-Galerkin method for the first order MFG system

Joint work with E. Carlini and F. J. Silva

First order MFG system

When $\nu = 0$ we have:

$$\begin{cases} -\partial_t u + H(x, Du) = F(x, m(t)) & \text{in } (0,T)\times\mathbb{R}^d, \\ \partial_t m - \operatorname{div}(D_p H(x, Du) m) = 0 & \text{in } (0,T)\times\mathbb{R}^d, \\ m(0,\cdot) = m_0^*, \quad u(T,x) = G(x, m(T)) & \text{in } \mathbb{R}^d. \end{cases} \qquad (\mathrm{MFG})_1$$

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- When the Hamiltonian H is coercive, the existence of solutions has been studied in Lasry-Lions'07 and in Cardaliaguet-Hadikhanloo'17.
- If H is not coercive, the existence question has been studied in Achdou-Mannucci-Marchi-Tchou'20 and in Cannarsa-Mendico'20.
- Unlike the second order case, solutions to $(\text{MFG})_1$ are not regular in general, which makes the analysis more complicated.

Numerical methods

- Coercive Case:
  - In Camilli-Silva'12, for $H(x,p) = |p|^2/2$, a semi-discrete SL scheme is proposed and convergence is shown.
  - A fully-discrete semi-Lagrangian proposed in Carlini-Silva'14, for $H(x,p) = |p|^2/2$, is shown to converge when $d = 1$.
  - Extensions to the case of fractional and non-local operators in Chowdhury-Ersland-Jakobsen'22.
  - Application to price formation MFG model by Ashrafyan-Gomes'24.
  - An approximating MFG with discrete time and finite state space is proposed in Hadikhanloo-Silva'19. Convergence is obtained in general dimensions.
  - Fourier methods, Nuberkyan, Saude ('19) and Liu, Jacobs, Li, Nuberkyan,Osher ('20)
- Non-coercive case:
  - See Gianatti-Silva'22 and Gianatti-Silva-Z'2023 where a relaxed definition of the equilibrium is used and an approximation based on discrete time finite state MFG is introduced.

Lagrange-Galerkin method for the first order MFG system

- **Main idea**: Apply a semi-Lagrangian scheme to the HJB equation then couple it with a Lagrange-Galerkin scheme for the continuity equation.

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- **Main idea**: Apply a semi-Lagrangian scheme to the HJB equation then couple it with a Lagrange-Galerkin scheme for the continuity equation.

- **Assumptions**:
  - The Hamiltonian H is given by

$$H(x,p) = \sup_{a \in \mathbb{R}^d} \{-\langle a,p\rangle - L(x,a)\} \quad \text{for all } x,p \in \mathbb{R}^d,$$

where L is of class C^2, and for all $x,a \in \mathbb{R}^d$, we have

$$\begin{aligned} L(x,a) &\le C(|a|^2+1), \\ |D_x L(x,a)| &\le C(|a|^2+1), \\ C|b|^2 &\le D^2_{aa} L(x,a)(b,b), \\ D^2_{xx} L(x,a)(y,y) &\le C(|a|^2+1)|y|^2. \end{aligned}$$

These assumptions on L imply that H has quadratic growth and

$$|D_p H(x,p)| \le C(1+|p|) \quad \text{for all } x,p \in \mathbb{R}^d.$$

A typical example is $H(x,p) = a(x)|p|^2 + \langle b(x),p\rangle$, with a and b of class C^2_b and a bounded from below by a strictly positive constant.

- F and G are bounded, continuous, and for every $\mu \in \mathcal{P}^1(\mathbb{R}^d)$,

$$\text{(Lip)} \quad |F(x,\mu) - F(y,\mu)| + |G(x,\mu) - G(y,\mu)| \leq C|x-y|,$$

$$\text{(SC)} \quad F(x+y,\mu) - 2F(x,\mu) + F(x-y,\mu) \leq C|y|^2,$$

$$\text{(SC)} \quad G(x+y,\mu) - 2G(x,\mu) + G(x-y,\mu) \leq C|y|^2.$$

  Notice that no differentiability is assumed for F and G.

- m_0^* has compact support and $m_0^* \in L^p(\mathbb{R}^d)$ for some $p \in (1,\infty]$.

Approximation to the HJB equation

Let $\mu \in C([0,T];\mathcal{P}_1(\mathbb{R}^d))$ and consider the HJB equation

$$-\partial_t u + H(x, Du) = F(x, \mu(t)) \quad \text{in } [0,T] \times \mathbb{R}^d,$$

$$u(T,x) = G(x, \mu(T)) \quad \text{in } \mathbb{R}^d.$$

If $u[\mu]$ denotes its solution, then for every $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$u[\mu](t,x) = \inf_{\alpha} \int_t^T \underbrace{L(\gamma(s), \alpha(s)) + F(\gamma(s), \mu(s))}_{\text{Running cost}} \, ds + \underbrace{G(\gamma(T), \mu(T))}_{\text{Final cost}}$$

$$\gamma \text{ satisfies } \dot{\gamma}(s) = -\alpha(s) \text{ in }]s,T[, \quad \gamma(t) = x.$$

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Proposition:

The value function is uniformly bounded, and the following hold:

$$\text{(Lip)} \quad |u[\mu](t,x) - u[\mu](t,y)| \le C|x-y|,$$

$$\text{(SC)} \quad u[\mu](x+y,\mu) - 2u[\mu](x,\mu) + u[\mu](x-y,\mu) \le C|y|^2.$$

Semi-Lagrangian scheme for HJB equation

- $u[\mu]$ satisfies the Dynamic Programming Principle:

$$u[\mu](t,x) = \inf_{\alpha \in L^2(\mathbb{R}^d)} \left\{ \int_t^{t+h} [L(\gamma(s),\alpha(s)) + F(\gamma(s),\mu(s)]\mathrm{d}s + u[\mu](t+h,\gamma(t+h)) \right\}$$

for all $h \in [0, T-t]$.

Semi-Lagrangian scheme for HJB equation

- Set $\Delta t > 0$ as the time step and let $t_k = k\Delta t$, $k = 0, \cdots N_T$.
- Semi-discrete DPP: let $u_k[\mu](x) \approx u[\mu](t_k, x)$ be such that

$$u_k[\mu](x) = \inf_{a \in \mathbb{R}^d} \Delta t [L(x, a) + F(x, \mu(t_k))] + u_{k+1}[\mu](x - a\Delta t)$$

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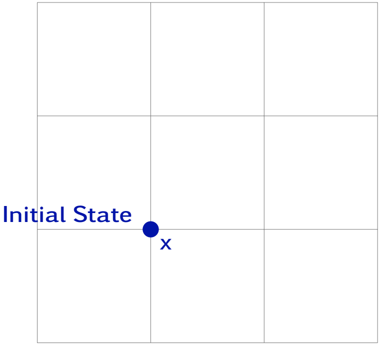
- Discretization in space: let $\Delta x > 0$ be the space step and let $\mathcal{G}_{\Delta x} = \{x_i = i\Delta x \,|\, i \in \mathbb{Z}^d\}$ be the grid space.



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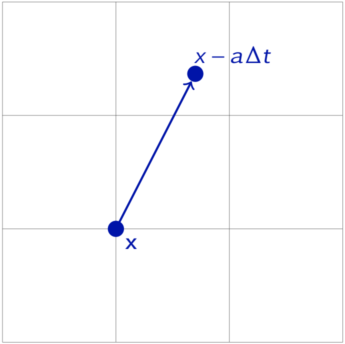

As in Carlini-S'14, given $(\Delta t, \Delta x)$ we consider the following **semi-Lagrangian** scheme for the HJB equation:

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\begin{aligned}
u_{k,i} &= \inf_{a \in \mathbb{R}^d} \left\{ \Delta t L(x_i, a) + I^1[u_{k+1,\cdot}](x_i - \Delta t a) \right\} + \Delta t F(x_i, \mu(t_k)), \\
u_{N,i} &= G(x_i, \mu(T)),
\end{aligned}
$$

where, given ϕ defined on $\mathcal{G}_{\Delta x} = \{x_i = \Delta x \,|\, i \in \mathbb{Z}^d\}$

$$
\mathrm{I}^1[\phi](x) = \sum_{i \in \mathbb{Z}^d} \beta_i^1(x) \phi(x_i), \quad \text{for all } x \in \mathbb{R}^d,
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where $\{\beta_i^1 \,|\, i \in \mathbb{Z}^d\}$ is the $\mathbb{Q}_1$-basis defined on the regular mesh $\mathcal{G}_{\Delta x}$.

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This scheme is shown to be consistent, stable, and preserves:

- (Lip) The Lipschitz property.
- (SC) The semiconcavity.

Given $\varepsilon > 0$ and a standard mollifier ρ_ε, we set $\Delta = (\Delta t, \Delta x, \varepsilon)$ and

$$u^\Delta[\mu](t,x) = (\rho_\varepsilon * I[u_k](x)) \quad \text{for all } t \in [t_k, t_{k+1}), x \in \mathbb{R}^d.$$

- $u^\Delta[\mu]$ preserves the Lipschitz property.
- The following semi-concavity estimate holds:

$$\langle D^2_{xx} u^\Delta[\mu](t,x)y, y\rangle \leq C\left(1 + \left(\frac{\Delta x}{\varepsilon^2}\right)^2\right)|y|^2.$$

- **Theorem:** Under suitable assumptions on the parameters, if $\mu_n \to \mu$ and $\Delta_n \to 0$, then $u^{\Delta_n}[\mu_n] \to u[\mu]$ uniformly over compact sets, and $D_x u^{\Delta_n}[\mu_n] \to D_x u[\mu]$ a.e.

Approximation to the continuity equation

Let us consider the following continuity equation

$$\partial_t m - \mathrm{div}(D_p H(x, D_x u[\mu])m) = 0 \quad \text{in } (0,T)\times \mathbb{R}^d,$$
$$m(0) = m_0^*.$$

Using the properties of $u[\mu]$, one can show the existence of $m[\mu]$ solution to the continuity equation such that:

- $m[\mu](t,\cdot)$ has a compact support, independent of μ.
- Mass conservation hold

$$\|m[\mu](t,\cdot)\|_{L^p} \le C\|m_0^*\|_{L^p}, \quad \text{for all } t \in (0,T).$$

where C is independent of p.

To discretize the MFG system, we focus on

$$\partial_t m - \mathrm{div}(D_p H(x, D_x u^{\Delta}[\mu])m) = 0 \quad \text{in } (0,T)\times\mathbb{R}^d,$$
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Since u^Δ is smooth w.r.t state, this equation has a unique solution

$$m^\Delta[\mu](t,\cdot) = \Phi^\Delta[\mu](0,t,\cdot)\sharp m_0^*,$$

where $\Phi^\Delta[\mu](s,t,x)$ is the solution, at time t, of the ODE:

$$\dot{\gamma}(r) = -D_p H(\gamma(r), D_x u^\Delta[\mu](r,\gamma(r))) \quad \text{in } (s,T),$$
$$\gamma(s) = x.$$

Equivalently, for ϕ integrable with respect to $m^\Delta[\mu](s)$,

$$\int_{\mathbb{R}^d} \phi(x)\, dm^\Delta[\mu](t)(x) = \int_{\mathbb{R}^d} \phi(\Phi^\Delta[\mu](s,t,x))\, dm^\Delta[\mu](s)(x) \tag{CE}$$

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- We approximate $\Phi^\Delta[\mu](t_k, t_{k+1}, x)$ by explicite one-step Euler scheme

$$\Phi_k^\Delta[\mu](x) = x - \Delta t D_p H(x, D_x u^\Delta[\mu](t_k, x)).$$

- Let $\{\beta_i\}_{i\in\mathbb{Z}^d}$ be a FE basis and approximate $m^\Delta[\mu](t_k)$ by

$$\mathbf{M}^\Delta[\mu](t_k, x) = \sum_{i\in\mathbb{Z}^d} m_{k,i}\beta_i(x).$$

- Using this approximation and taking $\phi = \beta_j$ in (CE), we get

$$\sum_{i\in\mathbb{Z}^d} m_{k+1,i}\int_{\mathbb{R}^d}\beta_i(x)\beta_j(\Phi_k^\Delta[\mu](x))\,\mathrm{d}x = \sum_{i\in\mathbb{Z}^d} m_{k,i}\int_{\mathbb{R}^d}\beta_j(\Phi_k^\Delta[\mu](x))\beta_i(x)\,\mathrm{d}x.$$

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- Let us choose $\beta_i=\beta_i^0=\mathbb{I}_{E_i}$, where

$$E_i=[x_i-\Delta x/2,x_i+\Delta x/2]^d.$$

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- Let us choose $\beta_i=\beta_i^0=\mathbb{1}_{E_i}$, where

$$E_i=[x_i-\Delta x/2,x_i+\Delta x/2]^d.$$

This choice yields the following Lagrange-Galerkin scheme:

$$m_{k+1,i}=\frac{1}{(\Delta x)^d}\sum_j m_{k,j}\int_{E_j}\beta_i^0(\Phi_k^\Delta[\mu](x))\,\mathrm{d}x \qquad \text{(LG)}$$

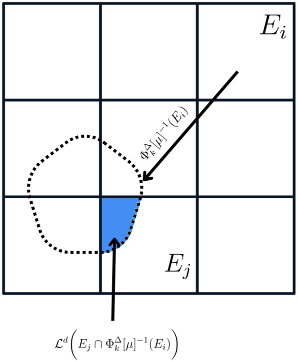
$$m_{0,i}=\frac{1}{(\Delta x)^d}\int_{E_i} m_0^*(x)\,\mathrm{d}x.$$

Interpretation of the scheme

- We observe that

$$\int_{E_j} \beta_i^0(\Phi_k^\Delta[\mu](x))\mathrm{d}x = \int_{\mathbb{R}^d} \mathbb{1}_{E_j\cap\Phi_k^\Delta[\mu]^{-1}(E_i)}(x)\mathrm{d}x = \mathcal{L}^d\left(E_j \cap \Phi_k^\Delta[\mu]^{-1}(E_i)\right),$$

$\Rightarrow$ equivalent to the scheme introduced in Picolli-Tosin'11.



Description of the scheme in the 2 dimensional case

- Given $(m_{k,i})$ solution to (LG), for $t \in [t_k, t_{k+1})$, let us define

$$\mathbf{M}^{\Delta}[\mu](t,x) = \left(\frac{t_{k+1}-t}{\Delta t}\right) \sum_{i \in \mathbb{Z}^d} m_{k,i}\beta_i(x) + \left(\frac{t-t_k}{\Delta t}\right) \sum_{i \in \mathbb{Z}^d} m_{k+1,i}\beta_i(x).$$

- $\mathbf{M}^{\Delta}[\mu] \in C([0,T];\mathcal{P}^1(\mathbb{R}^d))$.
- There exists $C^* > 0$ such that $\mathrm{supp}(\mathbf{M}^{\Delta}[\mu](t,\cdot)) \subseteq B(0,C^*)$.
- The map $[0,T] \ni t \mapsto \mathbf{M}^{\Delta}[\mu](t,\cdot) \in \mathcal{P}^1(\mathbb{R}^d)$ is Lipschitz continuous.
- If $\Delta x = O(\Delta t)$ and $\Delta t = O(\varepsilon^2)$, then

$$\|\mathbf{M}^{\Delta}[\mu](t,\cdot)\|_{L^p} \leq C\|m_0^*\|_{L^p}.$$

The proof of the L^p-stability mainly relies on the following facts:

- $\Delta t/\varepsilon$ small enough $\Rightarrow \Phi_k^{\Delta}[\mu]$ is one-to-one.
- The estimate on $D_{xx}^2 u^{\Delta}[\mu](t_k,\cdot)$ implies that

$$\det(D_x\Phi_k^{\Delta}[\mu](x))^{-1} \leq 1 + C\Delta t.$$

Approximation of the MFG problem

Let $u^{\Delta}[\mu]$ be the solution to the SL scheme and $\mathbf{M}^{\Delta}$ the solution to the LG scheme, then:

- (MFG)$_1$ is discretized as follows:

$$\text{Find } \mu \text{ such that } \mu = \mathbf{M}^{\Delta}[\mu] \qquad (\text{MFG})^{\Delta}.$$

  Using the Brouwer's fixed point theorem, we show that (MFG)$^{\Delta}$ admits at least one solution.

- Convergence holds in general state dimensions.

Theorem (Carlini-Silva-Z'23)

Let $\Delta_n = (\Delta t_n, \Delta x_n, \varepsilon_n) \in]0,\infty[^3$, let m^n be a solution to (MFG)$^{\Delta_n}$, and $u^n = u^{\Delta}[m_n]$. Assume that, as $\Delta_n \to 0$, $\Delta x_n = o(\Delta t_n)$ and $\Delta t_n = O(\varepsilon_n^2)$. Then, up to some subsequence, (u^n, m^n) converges to a solution (u^*, m^*) of (MFG)$_1$.

Numerical results

- In order to implement the scheme, we follow Morton-Priestley-Süli'88 by considering the following approximation called area weighting

$$\Phi_k^{\Delta}[\mu](x) \approx x - \Delta t D_p H(x_i, D_x v^{\Delta}[\mu](t_k, x_i)) \quad \text{if } x \in E_i,$$

to obtain

$$\int_{E_j} \beta_i^0(\Phi_k^{\Delta}[\mu](x)) \mathrm{d}x = \beta_i^1(\Phi_k^{\Delta}[\mu](x_j)).$$

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- We use Picard iterations to solve $(\mathrm{MFG})^{\Delta}$
- In the numerical test below, we set $d = 2$, and we consider the MFG problem defined on $[0,1] \times [0,2]^2$, and set $\Delta t = (\Delta x)^{\frac{2}{3}}$

$$m_0^*(x) = \frac{\nu(x)}{\int_{[0,2]^2} \nu(x)dx} \mathbb{I}_{[0,2]^2} \quad \text{with } \nu(x) = e^{-|x-x_0|^2/0.01} \quad \text{and } x_0 = (0.75, 0.75).$$

We also consider

$$H(x,p) = \frac{|p|^2}{2}, \quad G = 0$$

and

$$F(x,m) = \underbrace{\gamma \min(R, |x - x_f|^2)}_{\text{penalize the deviation from } x_f} + \underbrace{(\rho_\sigma * m)(x)}_{\text{encourage avoiding the crowd}}$$

with $x_f = (1.75, 1.75)$.

In the figures below, we display the distributions for $\gamma = 0.5$ and $\gamma = 3$.

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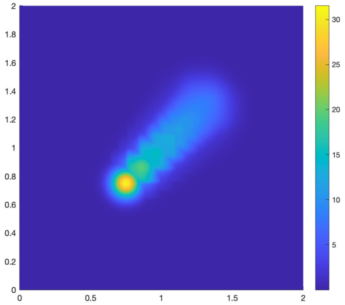

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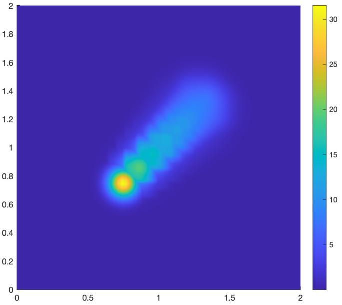

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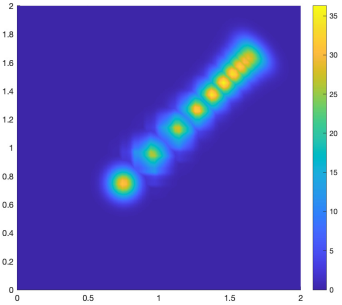

$\gamma = 3$

Newton iterations for second order MFG system

Ongoing work with E. Carlini and F. J. Silva

Newton iterations for second order MFG system

- We consider the second order MFG system

$$\begin{cases} -\partial_t u - \nu\Delta u + H(x, Du) = F(m(t,x)) & \text{in } \mathbb{T}^d \times [0,T] \\ \partial_t m - \nu\Delta m - \operatorname{div}(mH_p(x, Du)) = 0 & \text{in } \mathbb{T}^d \times [0,T] \\ m(x,0) = m_0(x),\ u(x,T) = u_T(x) & \text{in } \mathbb{T}^d\ , \end{cases} \qquad (\text{MFG})_2$$

  where
  - $\nu > 0$
  - $\mathbb{T}^d$ stands for the flat torus $\mathbb{R}^d/\mathbb{Z}^d$
  - H is a convex Hamiltonian
  - F is local coupling

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  where
  - $\nu > 0$
  - $\mathbb{T}^d$ stands for the flat torus $\mathbb{R}^d/\mathbb{Z}^d$
  - H is a convex Hamiltonian
  - F is local coupling
- **Our aim is:**
  to propose a new numerical scheme by discretizing a Newton method in infinite dimension

Numerical methods

- Y. Achdou, I. Capuzzo-Dolcetta ($'10$), Y. Achdou, F. Camilli, I. Capuzzo-Dolcetta ($'12$), *Semi-implicit finite difference scheme* computed through **Newton iterations**
- E. Carlini, F. J. Silva ($'14, '15$) *Semi-Lagrangian scheme* computed using **fixed point-type iterations**

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- Y. Achdou, I. Capuzzo-Dolcetta ('10), Y. Achdou, F. Camilli, I. Capuzzo-Dolcetta ('12), *Semi-implicit finite difference scheme* computed through **Newton iterations**
- E. Carlini, F. J. Silva ('14, '15) *Semi-Lagrangian scheme* computed using **fixed point-type iterations**
- H. Li, Y. Fan, and L. Ying ('21). *Multiscale method for mean field games. Second order accurate*
- S. Cacace, F. Camilli, A. Goffi ('23), Q. Tang, M. Lauriére ('23), *Policy iteration method.*
- Recent interest in machine learning techniques to solve $(\mathrm{MFG})_2$, e.g: deep learning, deep Galerkin method, reinforcement learning, etc..
- Summaries on numerical methods and learning methods for MFG: Y. Achdou, M. Laurière ('20) and M. Lauriére ('22).

Assumptions

Assumptions: For $\alpha \in (0,1)$:

1. m_0 is non-negative, $m_0 \in P(\mathbb{T}^d) \cap C^{2+\alpha}(\mathbb{T}^d)$, and $u_T \in C^{2+\alpha}(\mathbb{T}^d)$.
2. F, F', F'' are uniformly bounded mappings from $\mathbb{R}^+ \to \mathbb{R}$. Moreover, $F'(\cdot) \geq 0$.
3. $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is continuous, twice differentiable in p, and there exist constants $c, C > 0$ such that

$$cI \leq H_{pp}(x,p) \leq CI, \text{for all } (x,p) \in \mathbb{T}^d \times \mathbb{R}^d.$$

Under the above assumptions, $(\mathrm{MFG})_2$ admits one classical solution.

Newton method

- Following (Camilli Tang 2023) we define the map

$$\mathcal{F} : (u,m) \to \begin{pmatrix} -\partial_t u - \nu\Delta u + H(x,Du) - F(m) \\ \partial_t m - \nu\Delta m - \operatorname{div}(mH_p(x,Du)) \\ u(T) - u_T(x) \\ m(0) - m_0(x) \end{pmatrix}.$$

- Then

$$(\mathrm{MFG})_2 \Leftrightarrow \mathcal{F}(u,m) = 0.$$

- The corresponding Newton's iterations can be written as

$$J\mathcal{F}(u^{n-1},m^{n-1})((u^n,m^n) - (u^{n-1},m^{n-1})) = -\mathcal{F}(u^{n-1},m^{n-1}).$$

- Applying the Newton's iterations, we get the system

$$\begin{cases} -\partial_t u^n - \nu\Delta u^n + q^n Du^n = q^n Du^{n-1} - H(Du^{n-1}) + F(m^{n-1}) + F'(m^{n-1})(m^n - m^{n-1}) \\ \partial_t m^n - \nu\Delta m^n - \operatorname{div}(m^n q^n) = \operatorname{div}(m^{n-1} H_{pp}(Du^{n-1})(Du^n - Du^{n-1})) \\ m^n(x,0) = m_0(x), \quad u^n(x,T) = u_T(x) \end{cases} \qquad (\mathrm{MFG})_{\mathrm{NE}}$$

with $q^n = H_p(Du^{n-1})$.

Newton method

- The Newton methods reads:

$$\text{Given } (u^0, m^0), \text{ find } (u^n, m^n) \text{ by solving } (\text{MFG})_{\text{NE}} \text{ for } n \geq 1.$$

Theorem (Camilli Tang 2023)

If the initial guess (u^0, m^0) is close enough to the (u, m) solution of $(MFG)_2$, then

$$\|u^n - u\|_{C^{0,1}} + \|m^n - m\|_{C^0} \leq C(\|u^{n-1} - u\|_{C^{0,1}} + \|m^{n-1} - m\|_{C^0})^2.$$

- Notation:

$$\|u\|_{C^{0,1}} = \|u\|_{C^0} + \|Du\|_{C^0}$$

Our contribution

The question now is how to solve $(\text{MFG})_{\text{NE}}$

- For that we consider two different approaches
  1. An explicit semi-Lagrangian scheme
  2. An implicit upwind finite difference scheme
- A comparative analysis between the 2 aftermentioned schemes and other schemes from the literature.
- The comparison is based on the relative errors, number of iterations, CPU time and the robustness when $\nu \to 0$.
- For simplicity, we consider $d = 2$ and the quadratic Hamiltonian:

$$H(x,p) = \frac{|p|^2}{2} - V(x)$$

Main ingredients

- Given a grid function v, we introduce the first order central differences operators

$$
\begin{aligned}
(D_1 v)_{i,j} &= \frac{v_{i+1,j} - v_{i-1,j}}{2h} \quad i,j = 0, \cdots, N_h - 1, \\
(D_2 v)_{i,j} &= \frac{v_{i,j+1} - v_{i,j-1}}{2h} \quad i,j = 0, \cdots, N_h - 1,
\end{aligned}
$$

- The operator D_h as

$$
(D_h v)_{i,j} = ((D_1 v)_{i,j}, (D_2 v)_{i,j}) \quad i,j = 0, \cdots, N_h - 1.
$$

- The five point discrete Laplace operator:

$$
(\Delta_h v)_{i,j} = \frac{1}{h^2}(-4v_{i,j} + v_{i+1,j} + v_{i-1,j} + v_{i,j+1} + v_{i,j-1}) \quad i,j = 0, \cdots, N_h - 1.
$$

- Given a grid function with 2 components $q = (q_1, q_2)$, we define the discrete divergence operator

$$
\begin{aligned}
(\mathrm{div}_h(vq))_{i,j} = \frac{1}{2h}\Big(& v_{i+1,j}(q_1)_{i+1,j} - v_{i-1,j}(q_1)_{i-1,j} \\
& + v_{i,j+1}(q_2)_{i,j+1} - v_{i,j-1}(q_2)_{i,j-1}\Big).
\end{aligned}
$$

SL scheme for the backward equation

- Given

$$L^n(t,x) = \frac{|q^n(t,x)|^2}{2} + F(m^{n-1}(t,x)) + F'(m^{n-1}(t,x))(m^n(t,x) - m^{n-1}(t,x)) - V(x).$$

we consider

$$\begin{cases} -\partial_t u^n - \frac{\sigma^2}{2}\Delta u^n + q^n D u^n - L^n(t,x) = 0 & \text{in } [0,T]\times\mathbb{T}^2, \\ u^n(x,T) = G(x) & \text{in } \mathbb{T}^2, \end{cases}$$

with $\frac{\sigma^2}{2} = \nu$.

- Feynman-Kac formula

$$u^n(t,x) = \mathbb{E}\Big[\int_t^T L^n(s, X^{t,x}(s))\mathrm{d}s + G(X^{t,x}(T))\Big],$$

where $X^{t,x}$ denotes characteristics solving

$$\begin{cases} dX(s) = q^n(s,X(s)) + \sigma dW(s) & \text{for } s \in (t,T) \\ X(t) = x. \end{cases}$$

- Feynman-Kac formula in $[t_k, t_{k+1}]$

$$u^n(t_k,x) = \mathbb{E}\Big[\int_{t_k}^{t_{k+1}} L^n(s, X^{t_k,x}(s))\mathrm{d}s + u^n(t_{k+1}, X^{t_k,x}(\Delta t))\Big]$$

- Semi discretization in time by one-step weak Euler:

$$X^{t_k,x}(t_{k+1}) \approx x + \Delta t q^n(t_k,x) + \sigma\Delta W,$$

  where $P(\sigma\Delta W = \pm\sqrt{2\Delta t}) = \frac{1}{4}$

- Rectangular rule for running cost

$$\int_{t_k}^{t_{k+1}} L(s, X^{t_k,x})\mathrm{d}s \approx \Delta t L(t_k,x)$$

- Let us define $\{u_{i,j}^{n,k}\}$ as the solution to

$$\begin{cases} u_{i,j}^{n,k} = \dfrac{1}{4}\displaystyle\sum_{\ell=1}^{4} I[u^{n,k+1}]((x_{i,j} + \Delta t q^n(t_k, x_{i,j}) + \sqrt{2\Delta t}\sigma e^{\ell})_p) + \Delta t L^n(t_k, x_{i,j}), \\ u^{n,N_t} = u_T(x_{i,j}). \end{cases} \tag{SL}$$

Adjoint SL scheme for the forward equation

Given

$$G(t,x) = \mathrm{div}(m^{n-1}(t,x)(Du^n(t,x) - Du^{n-1}(t,x)))$$

let us consider

$$\begin{cases} \partial_t m^n - \frac{\sigma^2}{2}\Delta m^n - \mathrm{div}(m^n q^n) = G(t,x) & \text{in } [0,T]\times\mathbb{T}^2, \\ m^n(0,x) = m_0(x) & \text{in } \mathbb{T}^2. \end{cases}$$

Using the duality property

$$\int L(f) g \, \mathrm{d}x = \int L^*(g) f \, \mathrm{d}x$$

of the operators

$$L(u) := -\frac{\sigma^2}{2}\Delta u + q(x)^\top Du$$

$$L^*(m) := -\frac{\sigma^2}{2}\Delta m - \mathrm{div}(q(x)m)$$

we derive a scheme for the forward equation.

Adjoint SL scheme

- We define $\{m_{i,j}^{n,k}\}$ as solution to

$$\begin{cases} m_{i,j}^{n,k+1} = \dfrac{1}{4}\displaystyle\sum_{\ell=1}^{4} I^*[m^{n,k}](y_{i,j}^{\ell}(Q^{n,k})) + \Delta t(\mathrm{div}_h(m^{n-1,k+1}(D_h u^{n-1,k+1} - D_h u^{n,k+1})))_{i,j} \\ m_{i,j}^{n,0} = m_0(x_{i,j}), \end{cases} \tag{Adjoint-SL}$$

- $I^*[f](y_{i,j}^{\ell}(Q^{n,k}))$ is the adjoint operator of $f \to I[f](y_{i,j}^{\ell}(Q^{n,k}))$

Discrete Newton iterations system

- Denote by U and M vectors in $\mathbb{R}^{(N_t+1)N_h^2}$
- Combining (SL) and (Adjoint-SL), the semi-Lagrangian scheme to system $(\mathrm{MFG})_{\mathrm{NE}}$ can be written in a matrix form:

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  Given (U^{n-1}, M^{n-1}), define $Q^n := D_h U^{n-1}$ and compute (U^n, M^n) as solution of the Hamiltonian system

$$\begin{bmatrix} \mathbb{A} & -\mathbb{W} \\ -\mathbb{Z} & -\mathbb{A}^* \end{bmatrix} \begin{bmatrix} U \\ M \end{bmatrix} = \begin{bmatrix} \mathbb{b} \\ \mathbb{c} \end{bmatrix}. \qquad \text{(Newton-SL)}$$

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Proposition: If $M^n > 0$, then for any $n \in \mathbb{N}$ there exists a unique solution (U^n, M^n) to (Newton-SL).

Newton-finite differences scheme

- Given q^n, m^n, m^{n-1}, we define $\{u_{i,j}^{n,k}\}$ for $k = 0, \dots, N_{\Delta t} - 1$ as the solution to the following Implicit FD scheme:

$$\begin{cases} u_{i,j}^{n,k} = u_{i,j}^{n,k+1} + \Delta t \mu_{k,i} \Delta_h u_{i,j}^{n,k} + \Delta t q^n(t_k, x_{i,j}) D_h u_{i,j}^{n,k} + \Delta t L(t_k, x_{i,j}) \\ u_{i,j}^{n,N_{\Delta t}} = u_T(x_{i,j}). \end{cases}$$

  where

$$\mu_{i,j}^k = \nu + \frac{h}{2}(|q^n(t_k, x_{i,j})|)$$

- Computing the adjoint of the linearized backward equation to approximate the forward equation
- The Newton iteration system $(\mathrm{MFG})_{\mathrm{NE}}$ is approximated by

$$\begin{bmatrix} \mathbb{F} & -\tilde{\mathbb{W}} \\ -\tilde{\mathbb{Z}} & -\mathbb{F}^* \end{bmatrix} \begin{bmatrix} U \\ M \end{bmatrix} = \begin{bmatrix} \tilde{\mathbb{b}} \\ \tilde{\mathbb{c}} \end{bmatrix}, \qquad \text{(Newton-FD)}$$

Newton iteration algorithm for MFG

Algorithm Newton iterations for mean field games

Input: Initial guesses u^0, m^0, and tolerance τ
Output: Solution to the Newton iterations system $(\mathrm{MFG})_{\mathrm{NE}}$
$n \leftarrow 0$
repeat
    Compute m^{n+1} and u^{n+1} by Newton-SL or Newton-FD
    $\mathrm{err(m)} \leftarrow \|m^{n+1} - m^n\|_\infty$
    $\mathrm{err(u)} \leftarrow \|u^{n+1} - u^n\|_\infty$
    Update Q^n
    $n \leftarrow n + 1$
until $\mathrm{err(m)} < \tau$ **and** $\mathrm{err(u)} < \tau$
return m^{n+1}, u^{n+1}

Comparative analysis

- Through numerical tests, we conduct a comparative analysis between:
    1. Newton-SL
    2. Newton-FD
    3. FD-Newton (Achdou, Capuzzo-Dolcetta and Camilli. 2010)
    4. SL-FP (Carlini and Silva 2014)

Remark: In FD-Newton, a numerical Hamiltonian should be defined in order to get a discrete finite difference scheme for $(MFG)_2$, while in Newton-FD we only use central difference to discretize the Hamiltonian, which gives a simpler structure than FD-Newton.

Test 1: One dimensional MFG with a reference solution

- We consider a MFG system in the time-space domain $[0, 0.05] \times (0, 1)$ with periodic boundary conditions at $x = 0$ and $x = 1$, and $\nu = 0.1$.
- The Hamiltonian H is given by : $H(x, p) = \frac{|p|^2}{2}$
- The initial condition is given by

$$m_0(x) = \begin{cases} 4 \sin^2(2\pi(x - 1/4)) & \text{if } x \in [1/4, 3/4] \\ 0 & \text{otherwise,} \end{cases}$$

  and

$$F(m) = -3m_0(x) + 4 \min(4, m), \quad u_T(x) = 0, \quad \text{for } x \in (0, 1).$$

- The Newton stopping threshold is $\tau = 10^{-4}$.
- Reference solution to compare between the 4 schemes

Newton-SL vs SL-FP

Newton-SL				
h	$E_\infty(u)$	$E_\infty(m)$	Time	Iterations
$2.50 \cdot 10^{-2}$	$5.51 \cdot 10^{-2}$	$1.64 \cdot 10^{-1}$	0.61s	6
$1.25 \cdot 10^{-2}$	$2.40 \cdot 10^{-2}$	$1.16 \cdot 10^{-1}$	2.77s	7
$6.25 \cdot 10^{-3}$	$1.83 \cdot 10^{-2}$	$6.61 \cdot 10^{-2}$	13.92s	7
$3.125 \cdot 10^{-3}$	$4.50 \cdot 10^{-3}$	$1.41 \cdot 10^{-2}$	80.60s	7
SL-FP (Carlini and Silva'14)				
h	$E_\infty(u)$	$E_\infty(m)$	Time	Iterations
$2.50 \cdot 10^{-2}$	$5.75 \cdot 10^{-2}$	$1.62 \cdot 10^{-1}$	8.09s	10
$1.25 \cdot 10^{-2}$	$2.84 \cdot 10^{-2}$	$1.11 \cdot 10^{-1}$	40.79s	10
$6.25 \cdot 10^{-3}$	$2.15 \cdot 10^{-2}$	$5.84 \cdot 10^{-2}$	259.72s	12
$3.125 \cdot 10^{-3}$	$9.50 \cdot 10^{-3}$	$6.51 \cdot 10^{-3}$	2793.71s	12

Table: Errors for the approximation of solution (u, m) using Newton-SL and SL-FP.

Newton-FD vs FD-Newton

Newton-FD				
h	$E_\infty(u)$	$E_\infty(m)$	Time	Iterations
$2.50 \cdot 10^{-2}$	$1.532 \cdot 10^{-1}$	$3.42 \cdot 10^{-2}$	1.48s	7
$1.25 \cdot 10^{-2}$	$6.71 \cdot 10^{-2}$	$1.83 \cdot 10^{-2}$	12.27s	7
$6.25 \cdot 10^{-3}$	$3.37 \cdot 10^{-2}$	$9.51 \cdot 10^{-3}$	68.10s	7
$3.125 \cdot 10^{-3}$	$1.91 \cdot 10^{-2}$	$7.38 \cdot 10^{-3}$	436.01s	7
FD-Newton (Achdou et al.'13)				
h	$E_\infty(u)$	$E_\infty(m)$	Time	Iterations
$2.50 \cdot 10^{-2}$	$1.23 \cdot 10^{-1}$	$3.11 \cdot 10^{-2}$	2.23s	7
$1.25 \cdot 10^{-2}$	$6.21 \cdot 10^{-2}$	$1.63 \cdot 10^{-2}$	18.32s	8
$6.25 \cdot 10^{-3}$	$3.14 \cdot 10^{-2}$	$8.75 \cdot 10^{-3}$	92.91s	8
$3.125 \cdot 10^{-3}$	$1.77 \cdot 10^{-2}$	$9.54 \cdot 10^{-3}$	597.21s	8

Table: Errors for the approximation of solution (u, m) using FD-Newton and Newton-FD.

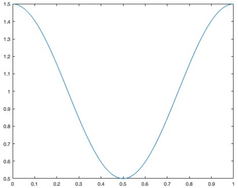
Test 2: One dimensional MFG

- We consider a MFG system in the time-space domain $[0, 0.01] \times]0, 1[$ with periodic boundary conditions.
- We vary the diffusion coefficient, taking values of $\nu = 0.4$ and $\nu = 0.02$.
- We consider the following data

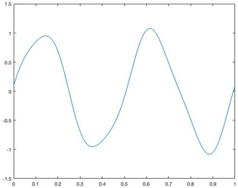
$$
\begin{array}{rcl}
m_0(x) & = & 1 + \frac{1}{2}\cos(2\pi x), \\
u_T(x) & = & \sin(4\pi x) + 0.1\cos(10\pi x), \\
H(x,p) & = & |p|^2 - V(x), \quad V(x) = 200\cos(2\pi x) - 10\cos(4\pi x), \\
F(m) & = & m^2.
\end{array}
$$

- The threshold τ for the Newton stopping iteration criteria is set to 10^{-4}
- Comparison between Newton-SL, Newton-FD and FD-Newton
- The results are coherent with H. Li, Y. Fan, and L. Ying ('21).

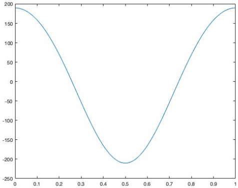
Test 2: $\nu = 0.4$



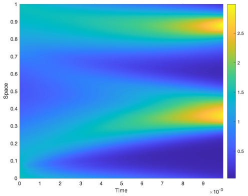
(a) Initial distribution m_0



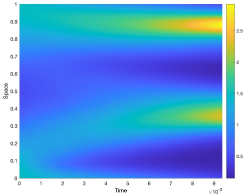
(b) Terminal cost u_T



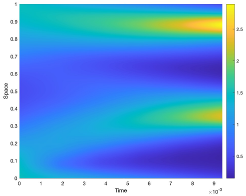
(c) Potential V



(a) Newton-SL



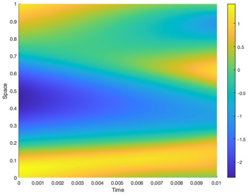
(b) Newton-FD



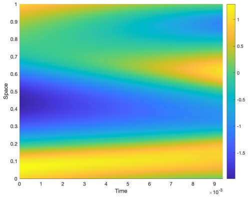
(c) FD-Newton

The distribution approximated with the three Newton schemes

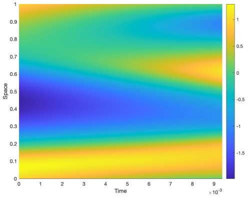
Test 2: $\nu = 0.4$



(a) Newton-SL

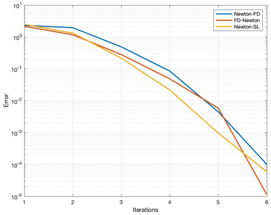


(b) Newton-FD

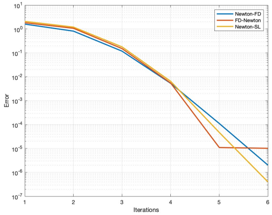


(c) FD-Newton

The value function approximated with the three Newton schemes



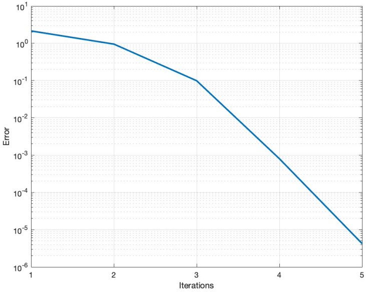
(a) $\|m^{n+1} - m^n\|_\infty$



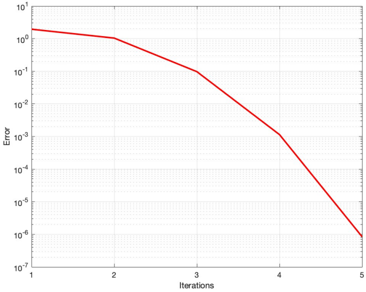
(b) $\|u^{n+1} - u^n\|_\infty$

Test 2: $\nu = 0.02$

- Breakdowns for Newton-FD and FD-Newton
- Newton-SL iterations error converge under the given threshold

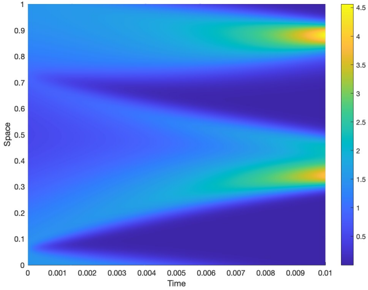


(a) $\|m^{n+1} - m^n\|_\infty$

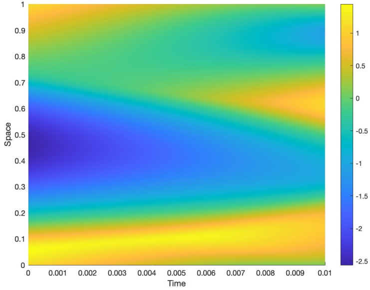


(b) $\|u^{n+1} - u^n\|_\infty$

Newton-SL iterations error for $\nu = 0.02$



(a) Evaluation of m



(b) Evaluation of u

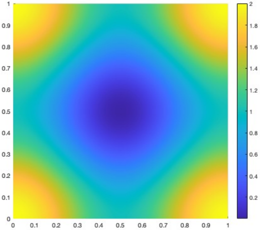
Approximated m and u using Newton-SL

Test 3: 2 dimensional MFG

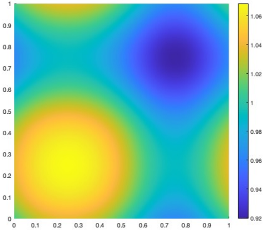
- We consider a MFG system in the time-space domain $[0,1]\times[0,1]^2$ with periodic boundary conditions.
- $\nu = 1$
- $H(x,y,p) = |p|^2 - V(x,y)$
- $\tau = 10^{-4}$
- We consider the following data

$$V(x,y) = cos(4\pi x) + sin(2\pi x) + sin(2\pi y), \quad F(m) = m^2,$$
$$m_0(x,y) = 1 + \frac{1}{2}cos(2\pi x) + \frac{1}{2}cos(2\pi y), \quad u_T(x,y) = cos(2\pi y) + cos(2\pi y).$$

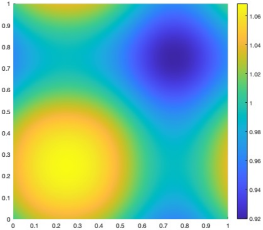
- We solve the MFG system using Newton-SL
- The results are coherent with H. Li, Y. Fan, and L. Ying ('21).



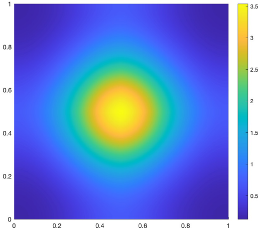
(a) $k = 0$



(b) $k = N_t/2$



(c) $k = 3N_t/4$



(d) $k = N_t$

The approximated distribution m at times $t = 0, \frac{N_t\,dt}{2}, \frac{3N_t\,dt}{4}, T$.

Conclusion

- FD-Newton and Newton-FD show similar behaviour in terms of CPU time and accuracy
- In our tests, Newton-SL needs the cheapest CPU time and shows comparable accuracy with respect to the other methods
- Newton-SL scheme works well in hyperbolic regime (ν small)

References

- E. Carlini and F. J. Silva, A fully discrete semi-Lagrangian scheme for a first order mean field game problem, SIAM Journal on Numerical Analysis, 52 (2014)
- E. Carlini, F. J. Silva, and A. Zorkot, A Lagrange–Galerkin scheme for first order mean field game systems, SIAM Journal on Numerical Analysis, 62 (2024)
- Y. Achdou and I. Capuzzo-Dolcetta. Mean field games: Numerical methods. SIAM Journal on Numerical Analysis, 48, 01 (2010).
- E.Carlini, F.J.Silva and A. Zorkot Newton methods for MFGs, in preparation (2024).

Thank you