Localized Inverse Design

Vincent Perrollaz

Institut Denis Poisson, Université de Tours

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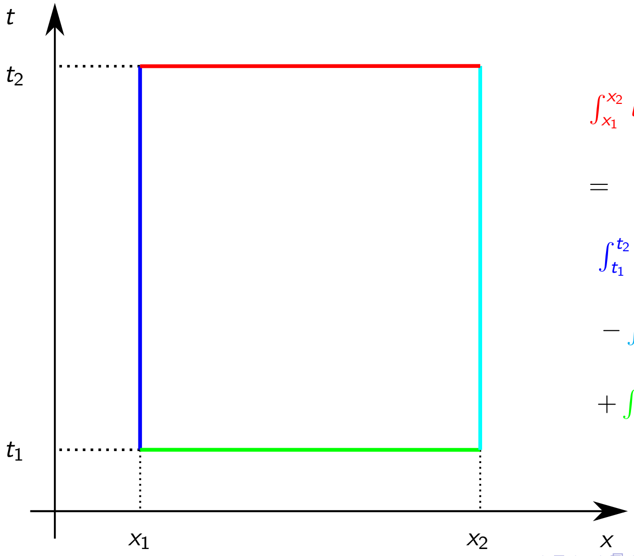
Integral Form



$$\int_{x_1}^{x_2} u(t_2, x)dx$$

$$=$$

$$\int_{t_1}^{t_2} F(t, x_1)dt$$

$$-\int_{t_1}^{t_2} F(t, x_2)dt$$

$$+\int_{x_1}^{x_2} u(t_1, x)dx$$

Differential Form

- Integral Form: u and F just L^1_{loc}.
- Differential Form:

$$
\begin{aligned}
& u, F \in \mathcal{C}^1 \\
& \qquad \Longrightarrow \int_{x_1}^{x_2} \int_{t_1}^{t_2} \partial_t u(t,x) dt dx = -\int_{t_1}^{t_2} \int_{x_1}^{x_2} \partial_x F(t,x) dx dt \\
& \qquad \Longrightarrow \frac{\partial u}{\partial t}(t,x) + \frac{\partial F}{\partial x}(t,x) = 0.
\end{aligned}
$$

Closure

- $F(t,x) = -\kappa \partial_x u(t,x)$ Heat Equation.

$$\partial_t u - \kappa \partial^2_{xx} u = 0.$$

- $F(t,x) = \frac{u^2(t,x)}{2}$ Burgers' equation (inspired by Euler's equation)

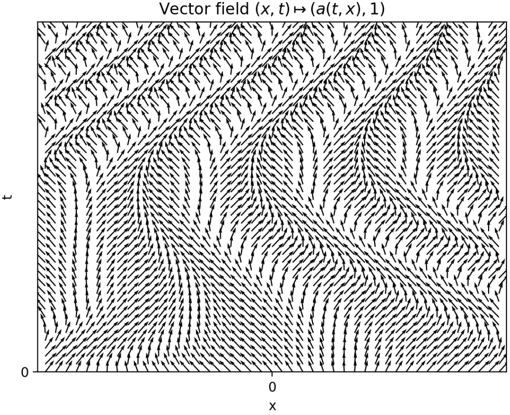
$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0$$

- $F(t,x) = u(t,x) v_{\max}(1 - \frac{u(t,x)}{u_{\max}})$ LWR equation

$$\partial_t u + \partial_x \left(u(t,x) v_{\max}(1 - \frac{u(t,x)}{u_{\max}}) \right) = 0$$

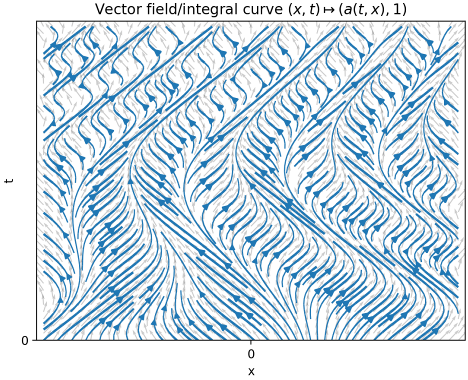
Characteristics' method I

$$\partial_t u + a(t,x)\partial_x u = 0 \iff (\partial_t + a(t,x)\partial_x)u = 0$$



Characteristics' method II

$$(\partial_t + a(t,x)\partial_x)u = 0 \iff \begin{cases} \frac{d}{dt}\psi(t,x) = a(t,\psi(t,x)), \\ \psi(0,x) = x \\ \frac{d}{dt}u(t,\psi(t,x)) = 0 \end{cases}$$



Characteristics' method III

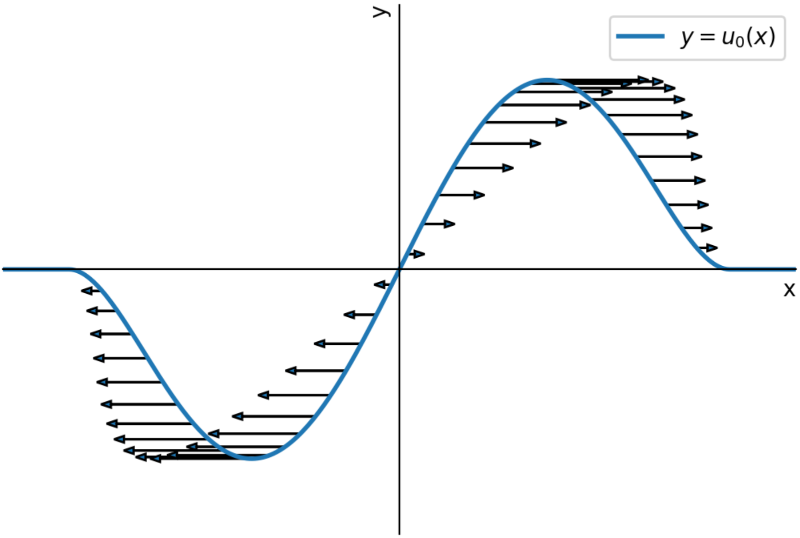
- $\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(0,x) = u_0(x) \\ u \in \mathcal{C}^1 \end{cases}$
- $\partial_t u + \partial_x f(u) = 0 \qquad \Longrightarrow \partial_t u + f'(u)\partial_x u = 0,$
- $\begin{cases} q \in \mathcal{C}^1(\mathbb{R}^+;\mathbb{R}) \\ p(t) := u(t,q(t)) \end{cases} \qquad \Longrightarrow \dot{p}(t) = \partial_t u + \dot{q}(t)\partial_x u,$

$$\text{“}\Longrightarrow\text{”} \begin{cases} \dot{q}(t) = f'(p(t)) \\ \dot{p}(t) = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} p(t) = p(0) \\ \dot{q}(t) = f'(p(0)) \end{cases}$$

$$\Longrightarrow u(t,x) = u_0(x - tf'(u(t,x)))$$

In a picture with Burgers $f'(p) = p$



Generic Blowup

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x). \end{cases} \tag{1}$$

Theorem

For $f(u) = \frac{u^2}{2}$ (in fact convex or concave) and **ANY** $u_0 \in \mathcal{C}_c^\infty(\mathbb{R})$

$$u_0 \not\equiv 0 \Rightarrow \exists T > 0, \quad \exists X \in \mathbb{R}, \qquad \partial_x u(t,X) \underset{t \to T^-}{\to} -\infty.$$

But

$$||u(t,\cdot)||_{L^\infty(\mathbb{R})} \leq ||u_0||_{L^\infty(\mathbb{R})}.$$

⚠ No linearization techniques!

Weak/Integral solutions

Three formulations, different regularity.

- Differential: $\partial_t u(t,x) + \partial_x f(u(t,x)) = 0,$

  $$\forall t > 0, \forall x \in \mathbb{R}$$

- Integral: $\dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\int_a^b u(t,x)dx = f(u(t,a)) - f(u(t,b))$

  $$\forall t > 0, \quad \forall a < b$$

- Weak: $\displaystyle\int_0^{+\infty} \int_{-\infty}^{+\infty} u(t,x)\partial_t\phi(t,x) + f(u(t,x))\partial_x\phi(t,x)dxdt = 0$

  $$\forall \phi \in \mathcal{C}_c^\infty((0,+\infty) \times \mathbb{R}).$$

Riemann initial data: one discontinuity!

- Simplest discontinuity and invariance by $x \mapsto x + \eta$:

$$u_0(x) := \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases}$$

- Invariance by $(t, x) \mapsto (\lambda t, \lambda x) \Rightarrow$

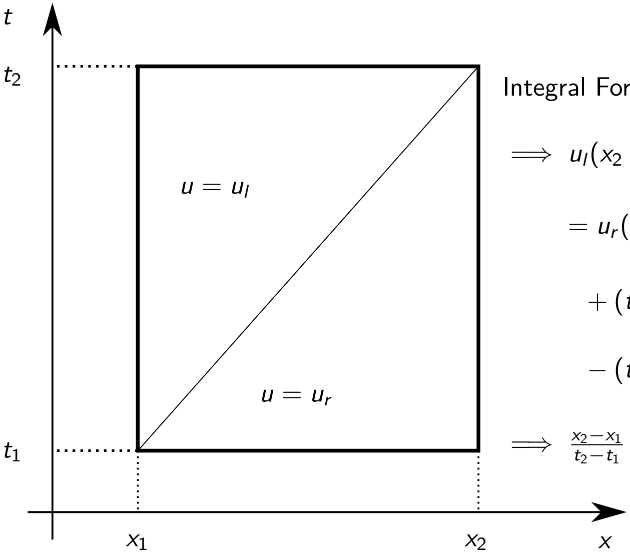
$$u(t, x) = v\left(\frac{x}{t}\right)$$

- Simplest case:

$$u(t, x) = \begin{cases} u_l & \text{if } x < \lambda t \\ u_r & \text{if } x > \lambda t \end{cases}$$

- What is λ?

Rankine-Hugoniot condition for weak solution



Integral Form

$$\implies u_l(x_2 - x_1)$$

$$= u_r(x_2 - x_1)$$

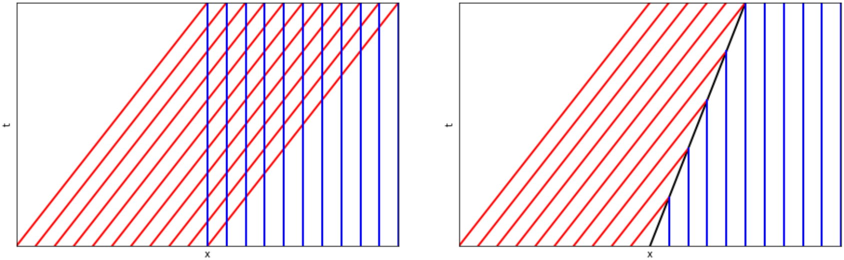
$$+ (t_2 - t_1) f(u_l)$$

$$- (t_2 - t_1) f(u_r)$$

$$\implies \frac{x_2 - x_1}{t_2 - t_1} = \frac{f(u_r) - f(u_l)}{u_r - u_l}$$

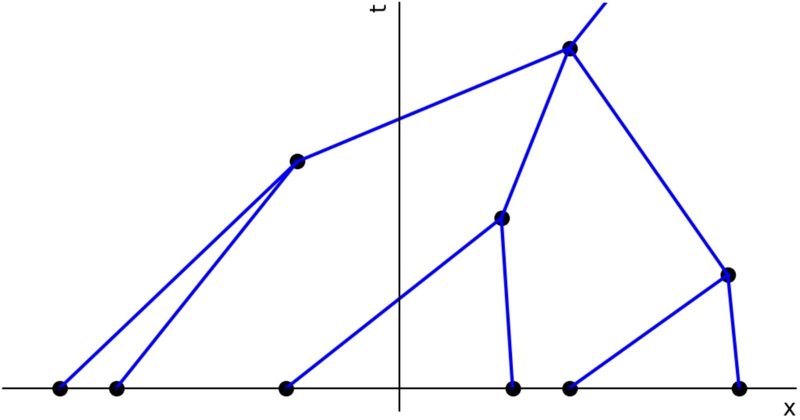
Burgers' characteristics: the Good

$$u_0(x) = \begin{cases} 1.0 & \text{if } x < 0 \\ 0.0 & \text{if } x > 0 \end{cases} \quad \Rightarrow \quad u(t,x) = \begin{cases} 1.0 & \text{if } x < \frac{t}{2} \\ 0 & \text{if } x > \frac{t}{2} \end{cases}$$



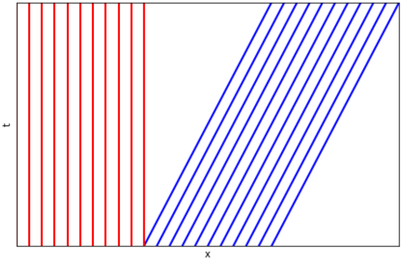
Semigroup?

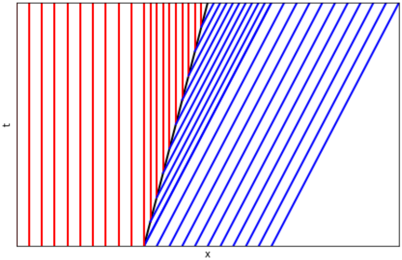
Propagation of discontinuities $\implies$ semigroup on piecewise constant functions!



Burgers' characteristics: the Bad

$$u_0(x) = \begin{cases} 0.0 & \text{if } x < 0 \\ 1.0 & \text{if } x > 0. \end{cases} \overset{?}{\Rightarrow} \quad u(t,x) = \begin{cases} 0 & \text{if } x < \frac{t}{2} \\ 1.0 & \text{if } x > \frac{t}{2} \end{cases}$$





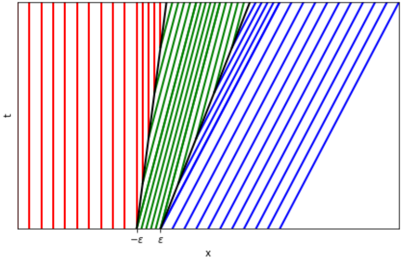
Continuous Semigroup?

- $\epsilon > 0$, consider

$$u_0^\epsilon(x) = \begin{cases} u_l & \text{if } x < -\epsilon \\ u_m & \text{if } -\epsilon < x < \epsilon \\ u_r & \text{if } \epsilon < x \end{cases}$$

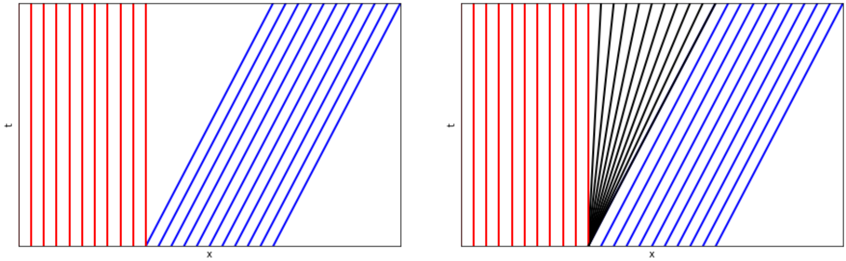
- $\epsilon \to 0$, admissibility condition on discontinuities (for continuous semigroup)

$$\forall u_m, \quad \frac{f(u_l) - f(u_m)}{u_l - u_m} \geq \frac{f(u_l) - f(u_r)}{u_l - u_r} \geq \frac{f(u_m) - f(u_r)}{u_m - u_r}$$



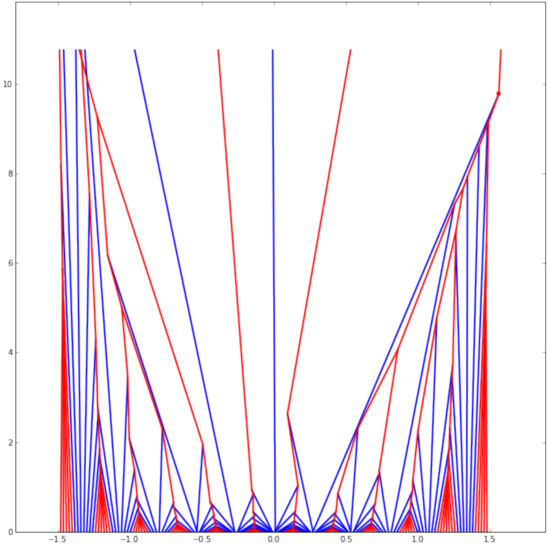
Rarefaction Wave

$$u_0(x) = \begin{cases} 0.0 & \text{if } x < 0 \\ 1.0 & \text{if } x > 0. \end{cases} \quad \Rightarrow \quad u(t,x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{t} & \text{if } 0 < x < t \\ 1.0 & \text{if } x > t \end{cases}$$



⚠ No reversibility in time.

Wave Front Tracking Algorithm (Dafermos, Holden-Risebro, Di Perna, Bressan)



Inverse Design Problem: Back to the Future!

- For this part: $u \mapsto f(u)$ strongly convex,
- Entropy solutions $\implies$ semigroup $(S_t^{CL})_{t\geq 0}$ acting on $L^\infty(\mathbb{R})$.
- Reachable states: given $T > 0$

$$\text{determine} \quad \{w \in L^\infty(\mathbb{R}) \,:\, \exists u_0 \in \mathrm{L}^\infty(\mathbb{R}) \quad S_T^{CL} u_0 = w\}.$$ [1]

- Inverse Design: given $T > 0$ and w in $\mathrm{L}^\infty(\mathbb{R})$

$$\text{determine} \quad I_T(w) := \{u_0 \in \mathrm{L}^\infty(\mathbb{R}) \,:\, S_T^{CL} u_0 = w\}.$$

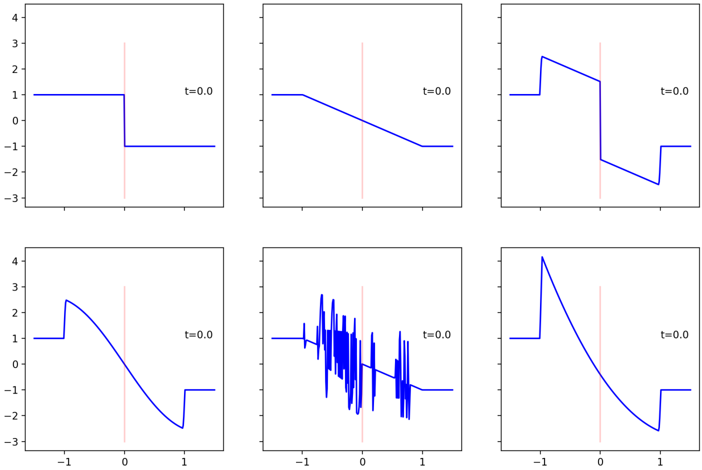
[1] Oleinik 57, Ancona-Marson 98, ...

Why?

1. Irreversible dynamics for entropy solutions.
2. Entropy semigroup compactifying[2].
3. Sonic boom minimization. (Gosse Zuazua 17)
4. Data assimilation for traffic flow through tollgate estimates, accident localization.
5. Control theory through Russell's extension method (Ancona-Marson 98, Horsin 98).

[2] De Lellis-Golse 2005, Ancona-Glass-Nguyen 2015, 2019, 2020

Burgers Slowdown: 10x



Characterization of reachable states

Going back to Oleinik 56!

Definition

$T > 0, \quad w \in L^\infty(\mathbb{R}) \qquad r_w^T(x) := x - Tf'(w(x)).$

Theorem

For f convex,

$$I_T(w) \neq \emptyset \iff w \in S_T^{CL}(L^\infty(\mathbb{R})) \iff r_w^T \text{ nondecreasing a.e.}$$

1. $w \in S_T^{CL}(L^\infty(\mathbb{R})) \implies w \in \mathrm{BV}(\mathbb{R})$,
2. Not better: take r_w^T Cantor's staircase.

Characterization of initial data

Theorem (Colombo-Perrollaz)

$I_T(w) \neq \emptyset \implies$

1. *$I_T(w)$ is a convex cone,*
2. *$I_T(w)$ is a F_σ set for the L^1_{loc} topology.*

Furthermore

1. *$I_T(w)$ singleton iff additionally $w \in \mathcal{C}^0$*
2. *otherwise unbounded L^∞, but locally L^∞ closed L^1_{loc},*
3. *and there is no extremal facet of finite dimension besides the vertex!*

In fact complete characterization of $I_T(w)$.

Tool: Hamilton-Jacobi connection AND optimal control

If $f^*(q) = \sup_{p \in R} (pq - f(p))$ (Legendre transform)

$$\begin{cases} V(t,x) := \inf_{c \in L^\infty(0,T)} \left(\int_0^t f^*(c(s))ds + P(y(0)) \right) \\ \dot{y}(s) = c(s) \quad 0 < s < t \\ y(t) = x \end{cases}$$

$$\iff \begin{cases} \partial_t V + f(\partial_x V) = 0, \\ V(0) = P \\ V \text{ viscosity solution} \end{cases}$$

$$\overset{u=\partial_x V}{\iff} \begin{cases} \partial_t u + \partial_x (f(u)) = 0, \\ u(0) = P' \\ u \text{ entropy solution} \end{cases}$$

+ specific minimizers for the optimal control problem!

Traffic Flow Problem

$$\begin{cases} \partial_\tau \rho + \partial_y(q(\rho)) = 0, \\ q(\rho(t,0)) = d_l(t) & t \in \mathbb{R}, \quad x \in (0,L), \\ q(\rho(t,L)) = d_r(t), \end{cases} \tag{2}$$

- Steady state evolution, **Not** a Cauchy problem!
- $L > 0$, ρ is a density (of cars for instance),
- q is bell shaped, $q(\rho) = v_{\max}(1 - \rho/\rho_{\max})\rho$ (LWR instance).
- ρ is an entropy solution,
- d_l and d_r are flow rates (for instance of cars through a toll gate)

Question: given $d_{r|[a,b]}$ what can we say about ρ?

Change of system

- **New quantity:** $u(t,x) := q(\rho(x,t))$
- **Hypothesis:** traffic is congested $q'(\rho(\tau,y)) < 0$ or not $q'(\rho(\tau,y)) > 0$ *everywhere at once!*
- **New flux:** f reciprocal of one of the branches of q, so convex!
- **Conclusion:** u is an entropy solution of

$$\partial_t u + \partial_x f(u) = 0, \tag{3}$$

satisfying

$$\begin{cases} u(0,x) = d_l(x), \\ u(L,x) = d_r(x). \end{cases} \tag{4}$$

- **Problem:** Inverse Design with localized knowledge of d_r in $[a,b]$ and u **has** to take value in $[\min q, \max q]$!

Back to the Future II

- J non trivial closed real interval,
- T positive number,
- K_0 non trivial closed real interval,
- K_T non trivial closed real interval.

Definition

A profile $u_T \in L^\infty(\mathbb{R}; J)$ is *reachable at* $t = T$ *on* K_T if there exists a $u_o \in L^\infty(\mathbb{R}; J)$ such that $S_T^{CL} u_{o|K_T} = u_T$.
Denote further

$$I_T(u_T; J)_{|K_o}^{|K_T} := \left\{ \tilde{u}_o \in L^\infty(K_o; J) \colon \exists\, u_o \in L^\infty(\mathbb{R}; J) \text{ with } \begin{array}{r} S_T^{CL} u_{o|K_T} = u_T \\ u_{o|K_o} = \tilde{u}_o \end{array} \right\}. \quad (5)$$

Characterization

Hypothesis:

- K_T non trivial compact interval,
- u_T in $L^\infty(\mathbb{R}, J)$ such that $r_T : x \mapsto x - f'(u_T(x))$ is nondecreasing on K_T,
- $K_0 := r_T(K_T)$, $\check{y} := \min K_0$.

Theorem (Colombo-Perrollaz)

There exist two functions $u_0^\flat$ and $u_0^\sharp$ in $L^\infty(K_0; J)$ such that

$$I_T(u_T; J)_{|K_o}^{|K_T} = \left\{u_o \in L^\infty(K_0; J) \ : \ \int_{\check{y}}^{y} u_o dx \in \left[\int_{\check{y}}^{y} u_o^\flat dx, \int_{\check{y}}^{y} u_o^\sharp dx\right] \ \textit{for all } y \in K_0\right\}$$

Remarks

- Direct characterization of $u_0^\flat$,
- Compactness of $I_T(u_T; J)_{|K_o}^{|K_T}$...

Remarks and Open Questions

- Variationnal formula for $u_0^\sharp$?
- Inverse Design for Sampling of the target profile?
- Propagation of measurement errors in the Inverse Design?
- Robust numerical methods for Inverse Design?
- Robustness using stochastic perturbation terms?
- Inverse Design for flux limited solutions of LWR model?

THANK YOU FOR YOUR ATTENTION