Trace of the gradient for HJB

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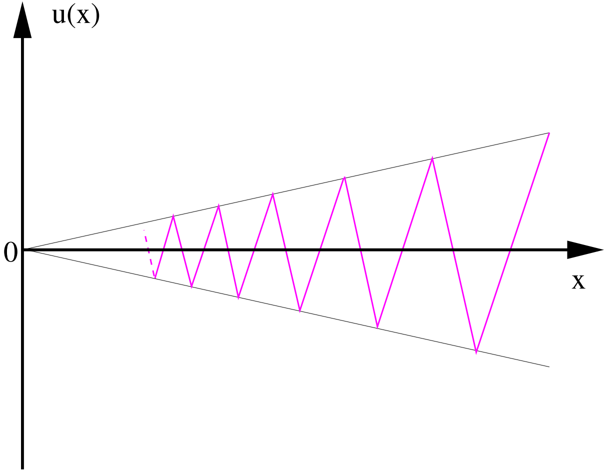
Main question

$$\left.\begin{array}{l} u \text{ viscosity solution of HJB} \\[2ex] ||Du||_{L^\infty} \le C \\[2ex] C^1 \text{ strictly convex Hamiltonian} \end{array}\right\} \quad \Longrightarrow \quad \text{trace of } Du\,?$$

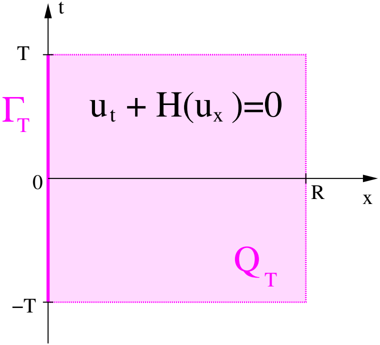
Case $H \equiv 0$ with no trace of u_x

$$u_t + H(u_x) = 0 \quad \text{for} \quad x > 0$$



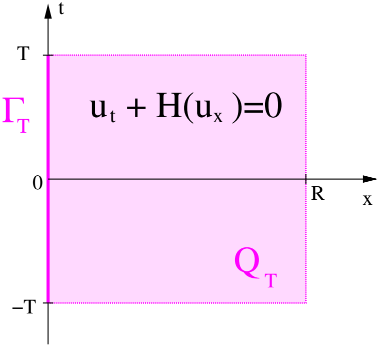
A model problem

A model problem in 1D



u Lipschitz up to the boundary Γ_T

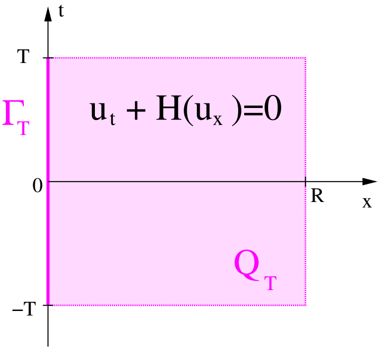
A model problem in 1D



$H : \mathbb{R} \to \mathbb{R}$ strictly convex

$$H(\lambda p + (1-\lambda)q) < \lambda H(p) + (1-\lambda)H(q) \quad \text{for all} \quad \lambda \in (0,1), \quad p \neq q$$

A model problem in 1D



$$\begin{cases} u \text{ Lipschitz up to the boundary } \Gamma_T, \\ H : \mathbb{R} \to \mathbb{R},\ C^1 \text{ strictly convex} \end{cases}$$

Can we define the trace of u_x on Γ_T ?

A good case on the whole line

$$\begin{cases} u_t + H(u_x) = 0 \quad \text{for} \quad t > 0, \quad x \in \mathbb{R} \\ H'' \geq 1/K \end{cases}$$

Then u is semiconcave with

$$u_{xx} \leq \frac{K}{t} \qquad \text{(Lax-Oleinik)} \qquad \Longrightarrow \qquad u_x \in BV_{t,x}$$

and u_x has a trace, say for $x = 0^+$.

See also
[Cannarsa, Sinestrari, 2004] : semiconcavity estimates
[Bianchini, De Lellis, Robyr, 2011] : $(u_t, u_x) \in SBV_{t,x}$

A bad case on the half line

$$\begin{cases} u_t + H(u_x) = 0 \quad \text{for} \quad t > 0, \quad x > 0 \\ H'' \geq 1/K \end{cases}$$

Then $|u_x|_{BV_x}$ can blow up in finite time.
[Adimurthi, Ghoshal, Dutta , Veerappa Gowda (2011)]

A model problem in 1D

$$\begin{cases} u_t + H(u_x) = 0 \quad \text{on the box} \quad Q_T := (-T,T) \times (0,R) \\ \\ u \text{ Lipschitz up to the boundary } \Gamma_T := (-T,T) \times \{0\} \\ H : \mathbb{R} \to \mathbb{R},\ C^1 \text{ strictly convex} \end{cases}$$

Again, can we define the trace of u_x on Γ_T ?

TTN result

$$\begin{cases} u_t + H(u_x) = 0 \quad \text{on the box} \quad Q_T := (-T,T) \times (0,R) \\ \\ u \text{ Lipschitz, up to the boundary } \Gamma_T := (-T,T) \times \{0\} \\ H : \mathbb{R} \to \mathbb{R},\ C^1 \text{ strictly convex} \end{cases}$$

Thm TTN (Tangential To Normal), [M.]

For $0 = (0,0) \in \Gamma_T$,

$$u_t(0) \text{ exists on } \Gamma_T \quad \Longrightarrow \quad u_x(0) \text{ exists on } \overline{Q}_T$$

TTN result

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Thm TTN (Tangential To Normal), [M.]

For $0 = (0,0) \in \Gamma_T$,

$$\lambda := u_t(0) \text{ exists on } \Gamma_T \quad \Longrightarrow \quad u_x(0) \text{ exists on } \overline{Q}_T$$

Precisely

$$u(t,0) - u(0,0) = \lambda t + o(|t|) \quad \Longrightarrow \quad u(X) - u(0) = \hat{D}u(0)\cdot X + o(|X|)$$

for $X = (t,x)$ and $\hat{D}u := (u_t, u_x)$.

Trace result

$$\begin{cases} u_t + H(u_x) = 0 \quad \text{on the box} \quad Q_T := (-T,T) \times (0,R) \\ \\ u \text{ Lipschitz on } \overline{Q}_T \\ H : \mathbb{R} \to \mathbb{R},\ C^1 \text{ strictly convex} \end{cases}$$

Thm 2 (Trace of the gradient), [M.]

The gradient $u_x(t,0)$ exists for a.e. $t \in (-T,T)$ and

$$\lim_{\varepsilon \to 0^+} \int_{Q_T} |u_x(t,\varepsilon x) - u_x(t,0)| \, dtdx = 0$$

Trace result

$$\begin{cases} u_t + H(u_x) = 0 \quad \text{on the box} \quad Q_T := (-T,T) \times (0,R) \\ \\ u \text{ Lipschitz on } \overline{Q}_T \\ H : \mathbb{R} \to \mathbb{R},\ C^1 \text{ strictly convex} \end{cases}$$

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$$\lim_{\varepsilon \to 0^+} \int_{Q_T} |u_x(t,\varepsilon x) - u_x(t,0)| \, dtdx = 0$$

and more generally

$$\hat{D}u(\cdot,\varepsilon\cdot) \to (\hat{D}u)_{|\Gamma_T} \quad \text{in} \quad L^1(\text{normal}; L^1(\text{tangential}))$$

(Pointwise to L^1, using Young measures).

Application : HJ to CL

If u is a Lipschitz viscosity solution of

$$\begin{cases} u_t + H(u_x) = 0 & \text{for} \quad x > 0 \\ -u_x + v_D = 0 & \text{for} \quad x = 0 \qquad \text{(weakly)} \end{cases}$$

then $v := u_x$ is an entropy solution of

$$\begin{cases} v_t + H(v)_x = 0 & \text{for} \quad x > 0 \\ v = v_D & \text{for} \quad x = 0 \qquad \text{(weakly)} \end{cases}$$

See also (when moreover $H'' \geq \delta > 0$)
[Cardaliaguet, Forcadel, M., 2023]
[Cardaliaguet, Forcadel, Girard, M., 2023]
[Imbert, Forcadel, M., 2023] : general boundary conditions

For

$$v_t + H(v)_x = 0 \quad \text{for} \quad x > 0$$

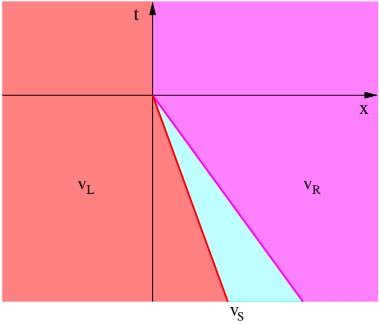
Trace of v on $\{x = 0\}$ for "nonlinear H" :
[Vasseur, 2001]
[Panov, 2007]

A counter-example to TTN result, with H not convex

There exists a viscosity solution of $u_t + H(u_x) = 0$ with $u(t,0) \equiv 0$ and

$$v_S < v_R < v_L, \qquad C^{1,1}([v_S, v_L]) \ni H = \begin{cases} \text{concave on } [v_S, v_R] \\ \text{convex on } [v_R, v_L] \end{cases}$$

and $v := u_x$ entropy solution of $v_t + H(v)_x = 0$



Here $u(X) - u(0) \neq \hat{D}u(0) \cdot X + o(|X|)$.

Higher dimensions and (t, x)-dependence

$\Omega \subset \mathbb{R}^n$ and X-dependence

$$\begin{cases} u_t + H(Du, X) = 0 & \text{on} \quad Q_T := (0,T) \times \Omega \ \ni (t,x) = X \\ u \text{ Lipschitz on } \overline{Q}_T, & H : \mathbb{R}^n \times \overline{Q}_T \to \mathbb{R} \text{ continuous} \\ H : p \mapsto H(p, X_0),\ C^1 \text{ strictly convex at some } X_0 \in \Gamma_T := (0,T) \times \partial\Omega & \end{cases}$$

Thm TTN' (Tangential To Normal), [M.]

Let

$$\hat{D}u = (u_t, Du)$$

If $\partial\Omega \in C^1$, then

$$\hat{D}(u_{|\Gamma_T})(X_0) \text{ exists} \quad \Longrightarrow \quad \hat{D}u(X_0) \text{ exists}$$

$\Omega \subset \mathbb{R}^n$ and X-dependence

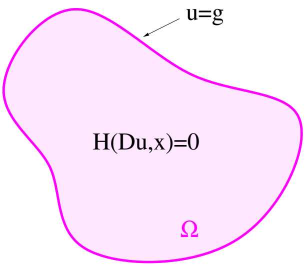
$$\begin{cases} u_t + H(Du, X) = 0 \quad \text{on} \quad Q_T := (0,T) \times \Omega \ni (t,x) = X \\ \qquad \text{with the cylinder} \quad \Omega := B_1' \times (0,1) \ni (x', x_n) \\ u \text{ Lipschitz on } \overline{Q}_T, \qquad H : \mathbb{R}^n \times \overline{Q}_T \to \mathbb{R} \text{ continuous} \\ H : p \mapsto H(p, X_0),\ C^1 \text{ strictly convex at all } X_0 \in \Gamma_T := (0,T) \times B_1' \times \{0\} \end{cases}$$

Thm 2' (Trace of the gradient), [M.]

Then $\hat{D}u := (u_t, Du)$ exists a.e. on Γ_T and

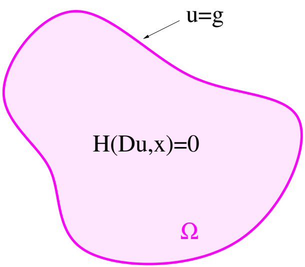
$$\lim_{\varepsilon \to 0^+} \int_{Q_T} |\hat{D}u(t, x', \varepsilon x_n) - \hat{D}u(t, x', 0)| \, dt dx' dx_n = 0$$

Application : Dirichlet problem



$H : p \mapsto H(p, x_0)$, C^1 strictly convex for all $x_0 \in \partial\Omega$

Application : Dirichlet problem

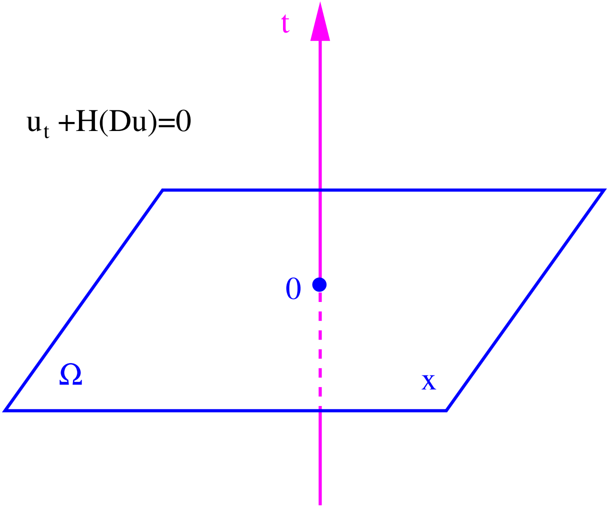


$H : p \mapsto H(p, x_0)$, C^1 strictly convex for all $x_0 \in \partial\Omega$

Thm TTN* (Existence of a normal derivative), [M.]

If u is Lipschitz on $\overline{\Omega}$, and $\partial\Omega \in C^1$ with $g \in C^1(\partial\Omega)$,
then $\frac{\partial u}{\partial n}$ exists everywhere on $\partial\Omega$.

Codimension $d > 1$ and (t, x)-dependence



Similar to
[Achdou, Le Bris, 2023]

$$\begin{cases} u_t + H(Du) = 0 \quad \text{on} \quad Q_T := (-T,T) \times \Omega, \quad \Omega := \mathbb{R}^d \setminus \{0\} \\ \qquad\qquad\qquad\quad \text{with} \quad \Gamma_T := (-T,T) \times \{0\} \\ \\ u \text{ Lipschitz on } \overline{Q}_T, \quad H : \mathbb{R}^d \to \mathbb{R},\ C^1 \text{ strictly convex} \end{cases}$$

Thm TTN"' (Tangential To Normal), [M.]

For $0 = (0,0) \in \Gamma_T$ and $\hat{D}u = (u_t, Du)$, we have

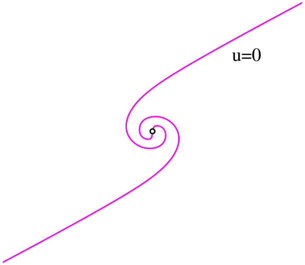
$$\hat{D}(u_{|\Gamma_T})(0) \quad \text{exists along } \Gamma_T \quad \Longrightarrow \quad u \text{ has all direct. deriv. at } 0 \text{ along } \overline{Q}_T$$

$$\frac{u(\varepsilon X) - u(0)}{\varepsilon} =: u^\varepsilon(X) \to u^0(X) \quad \text{with} \quad u^0(\lambda X) = \lambda u^0(X), \quad \lambda > 0$$

A counter-example with $H(p,x)$ not Hölder in x

For $\partial_t = 0$, $d = 2$, there exists a viscosity solution u of

$$\begin{cases} a(x)|Du|^2 = 1 \quad \text{on} \quad B_1 \subset \mathbb{R}^2 \\ 1 \le a \le 2, \quad \text{with } a \text{ only continuous on } B_1 \end{cases}$$



and

$$\frac{u(\varepsilon x) - u(0)}{\varepsilon} =: u^\varepsilon(x) \quad \text{does not converges as } \varepsilon \to 0$$

Codimension $d > 1$, Hölder coefficients

$$\begin{cases} u_t + H(Du, X) = 0 \quad \text{on} \quad Q_T := (-T,T) \times B_1' \times \Omega, \qquad B_1' \subset \mathbb{R}^m \\ \Omega := B_1 \setminus \{0\} \subset \mathbb{R}^d \quad \text{with} \quad \Gamma_T := (-T,T) \times B_1' \times \{0\} \\ \\ u \text{ Lipschitz on } \overline{Q}_T, \quad H : \mathbb{R}^{m+d} \times \overline{Q}_T \to \mathbb{R} \text{ continuous} \end{cases}$$

$$\left.\begin{array}{l} 0 < \delta \leq D^2_{pp} H \leq \delta^{-1} \\ H \ \alpha\text{-Hölder in } X \text{ for bounded } p \end{array}\right\} \quad \text{for all } X \text{ in a neighborhood of } 0$$

Thm TTN'''' (Tangential To Normal), [M.]

For $0 = (0,0,0) \in \Gamma_T$ and $\hat{D}u = (u_t, Du)$, we have

$$\hat{D}(u_{|\Gamma_T})(0) \quad \text{exists along } \Gamma_T \quad \Longrightarrow \quad u \text{ has all direct. deriv. at } 0 \text{ along } \overline{Q}_T$$

$$\frac{u(\varepsilon X) - u(0)}{\varepsilon} =: u^\varepsilon(X) \to u^0(X) \quad \text{with} \quad u^0(\lambda X) = \lambda u^0(X), \quad \lambda > 0$$

Codimension $d > 1$, Hölder coefficients

same assumptions

$$\left.\begin{array}{l} 0 < \delta \leq D^2_{pp}H \leq \delta^{-1} \\ H\ \alpha\text{-Hölder in } X \text{ for bounded } p \end{array}\right\} \quad \text{for all } X \text{ in a neighborhood of } \Gamma_T$$

Thm 2'''' (Trace of directional derivatives), [M.]

Let $\hat{D}u = (u_t, Du)$. Then a.e. on Γ_T and for any $\xi \in \mathbb{R}^{1+m+d}$, the quantity $\xi \cdot \hat{D}u$ exists and

$$\lim_{\varepsilon \to 0^+} \int_{Q_T} \left|\xi \cdot \hat{D}u(t, x', \varepsilon x) - \xi \cdot \hat{D}u(t, x', 0)\right| \, dt dx' dx = 0$$

Arguments for TTN

A Liouville-type result

$$\begin{cases} u_t + H(Du) = 0 & \text{on} \quad Q = \mathbb{R} \times \mathbb{R}^m \times \Omega, \\ u = 0 & \text{on} \quad \Gamma := \mathbb{R} \times \mathbb{R}^m \times \{0\} \end{cases}$$

with

$$\Omega := \begin{cases} \mathbb{R}^d \setminus \{0\} & \text{if} \quad d \geq 2 \\ (0, +\infty) & \text{if} \quad d = 1 \end{cases}$$

Thm 4 (Liouville-type result), [M.]

Let $H : \mathbb{R}^{m+d} \to \mathbb{R}$ be C^1 strictly convex and superlinear. If u is a viscosity solution and $||(u_t, Du)||_{L^\infty(\overline{Q})} < +\infty$, then

$$u(t, x', x) = u(0, 0, x) \quad \text{for all} \quad (t, x', x) \in \mathbb{R} \times \mathbb{R}^m \times \overline{\Omega}$$

Consequence for the first blow-up

If

$$\begin{cases} u_t + H(Du, X) & = 0 \quad \text{on} \quad Q_T, \\ u & = 0 \quad \text{on} \quad \Gamma_T \ni 0 \end{cases}$$

then

$$\frac{u(\varepsilon X) - u(0)}{\varepsilon} =: u^\varepsilon(X) \to u^0(X)$$

where the blow-up limit u^0 is a global solution of

$$\begin{cases} u^0_t + H(Du^0, 0) & = 0 \quad \text{on} \quad Q = \mathbb{R} \times \mathbb{R}^m \times \Omega, \\ u^0 & = 0 \quad \text{on} \quad \Gamma := \mathbb{R} \times \mathbb{R}^m \times \{0\} \end{cases}$$

Case $m = 0$, codimension $d = 1$, without X

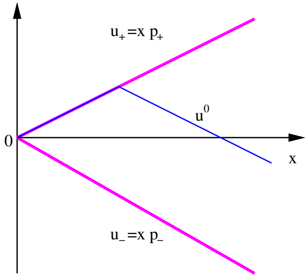
$$\begin{cases} u^0_t + H(u^0_x) &= 0 \quad \text{on} \quad Q := \mathbb{R} \times (0, +\infty), \\ u^0 &= 0 \quad \text{on} \quad \Gamma := \mathbb{R} \times \{0\} \end{cases}$$

Then

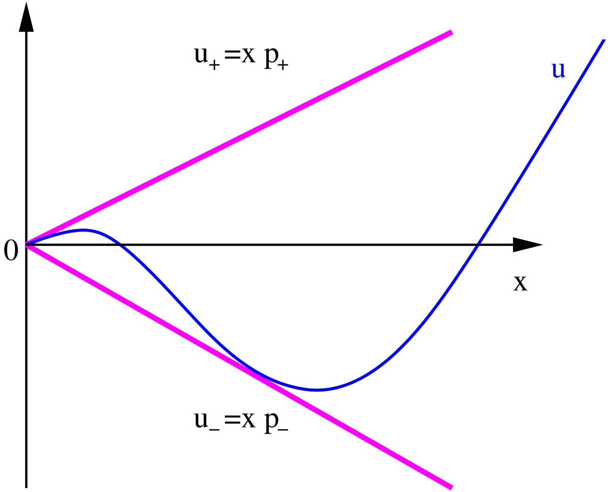
$$\text{Liouville} \quad \Longrightarrow \quad u^0(t,x) = u^0(x)$$

and

$$u^0(x) = \min(xp_+, c + xp_-) \quad \text{with} \quad H(p_\pm) = 0, \quad p_- \leq p_+$$



Exclusion of oscillations of u between u_- and u_+



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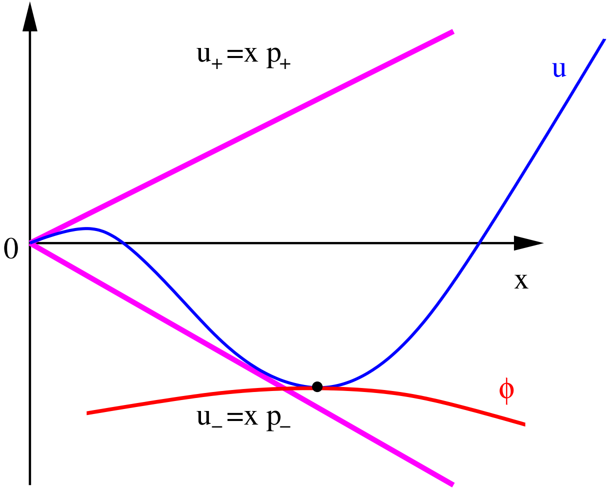


$0 + H(\phi_x) \geq 0$ contradiction with $\begin{cases} p_- < 0 < p_+ \\ H(p_\pm) = 0 > H(0) \end{cases}$

Conclusion

Uniqueness of the limit $\quad u^{\varepsilon} \to u^0 \in \{u_-, u_+\}$

Hence

$$\frac{u(\varepsilon X) - u(0)}{\varepsilon} \to u^0(x) = px \quad \text{with} \quad p = p_\pm$$

Conclusion

Uniqueness of the limit $u^{\varepsilon} \to u^0 \in \{u_-, u_+\}$

Hence

$$\frac{u(\varepsilon X) - u(0)}{\varepsilon} \to u^0(x) = px \quad \text{with} \quad p = p_\pm$$

i.e.

$$u(X) = u(0) + px + o(|X|) \quad \text{where} \quad X = (t, x)$$

Hence

$$\hat{D}u = (u_t, u_x) \quad \text{exists at } X = (0,0) = 0$$

Arguments for Liouville result

Step 1 : barriers $u_\pm$

$$\begin{cases} u_t + H(Du) = 0 & \text{on} \quad Q = \mathbb{R} \times (0, +\infty), \\ u = 0 & \text{on} \quad \Gamma = \mathbb{R} \times \{0\} \end{cases}$$

with

$$||(u_t, u_x)||_{L^\infty(\overline{Q})} < +\infty$$

By comparison, we show

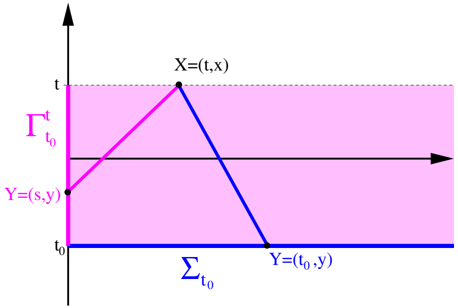
$$u_-(x) \le u(t,x) \le u_+(x), \quad u_\pm(x) = xp_\pm$$

Define the characteristic velocities

$$\xi_\pm := DH(p_\pm) \quad \Longrightarrow \quad \xi_- < 0 < \xi_+$$

For $t_0 < t$ and $x > 0$, we have

$$u(X) := \min(u_{\Gamma_{t_0}^t}, u_{\Sigma_{t_0}})(X)$$



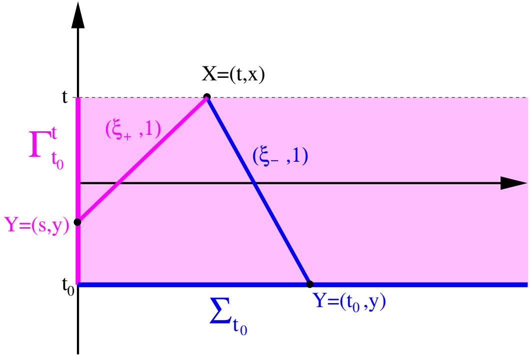
with $X = (t,x)$, $Y = (s,y)$ and

$$u_A(X) := \inf_{Y \in A} \left\{ u(Y) + \int_s^t \mathcal{L}\left(\frac{x-y}{t-s}\right) d\tau \right\} \quad \text{with} \quad \mathcal{L} := H^*$$

Step 3 : Rigidity of optimal trajectories

For $t_0 < t$ and $x > 0$, we have

$$u(X) := \min(u_{\Gamma^t_{t_0}}, u_{\Sigma_{t_0}})(X)$$



with

$$\xi_-, \xi_+ \quad \text{selected by the barriers } u_-, u_+$$

Step 3 : Rigidity of optimal trajectories

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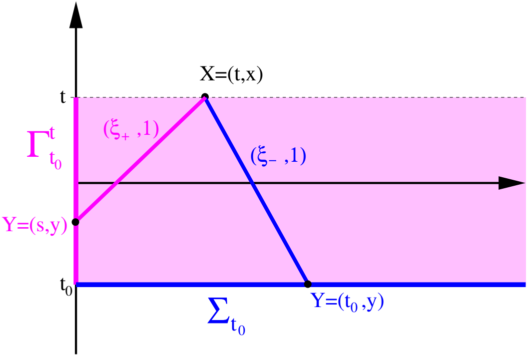


This forces

$$u_{\Gamma^t_{t_0}}(X) = u_+(x), \quad u_{\Sigma_{t_0}}(X) = c + u_-(x) \quad \text{for} \quad \frac{t - t_0}{x} >> 1$$

Comments on the general case

Codimension $d > 1$ is much more delicate.

X-dependence :
need to control $\xi_-(X)$ which can rotate.

Conjecture

$$u_t + H(Du) = 0 \quad \text{in} \quad \{x_n > 0\} \tag{1}$$

Trace conjecture

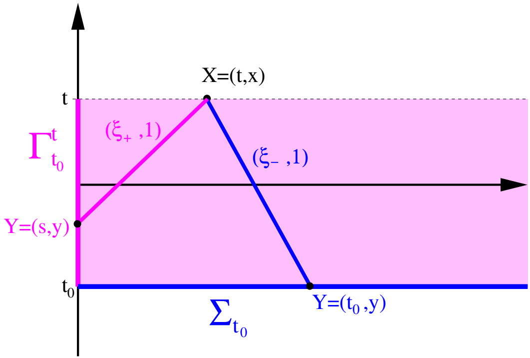
Let u be a Lipschitz viscosity solution of (1).
If the graph of H does not contain any nondegenerate segments,
then (u_t, Du) has a trace on $\{x_n = 0\}$.

THANKS !

The End

For $t_0 < t$ and $x > 0$, we have

$$u(X) := \min(u_{\Gamma_{t_0}^t}, u_{\Sigma_{t_0}})(X)$$



with

$$u_{\Gamma_{t_0}^t}(X) = u_+(x), \quad u_{\Sigma_{t_0}}(X) = c + u_-(x) \quad \text{for} \quad \frac{t-t_0}{x} >> 1$$

Step 3 : Explanation

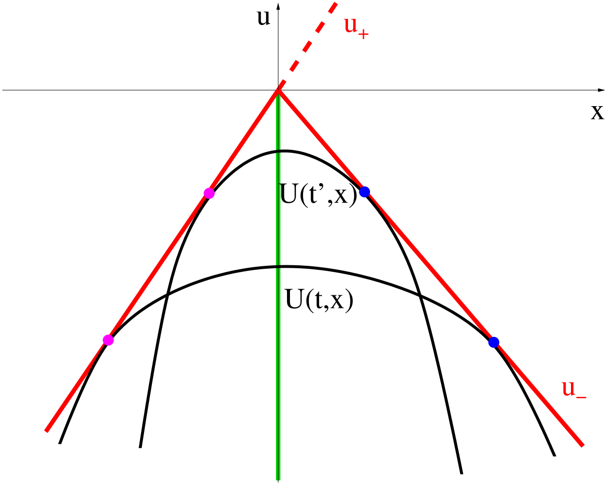
For $\mathcal{L} = H^*$,

$$U(t,x) := \begin{cases} t\mathcal{L}\left(\frac{x}{t}\right) & \text{for} \quad (t,x) \in (-\infty, 0) \times \mathbb{R} \\ 0 & \text{for} \quad t = 0, \quad x = 0 \\ -\infty & \text{for} \quad t = 0, \quad x \neq 0 \end{cases}$$

is a fundamental solution of

$$u_t + H(Du) = 0 \quad \text{on} \quad (-\infty, 0) \times \mathbb{R}$$

Step 3 : U tangential to $u_\pm$

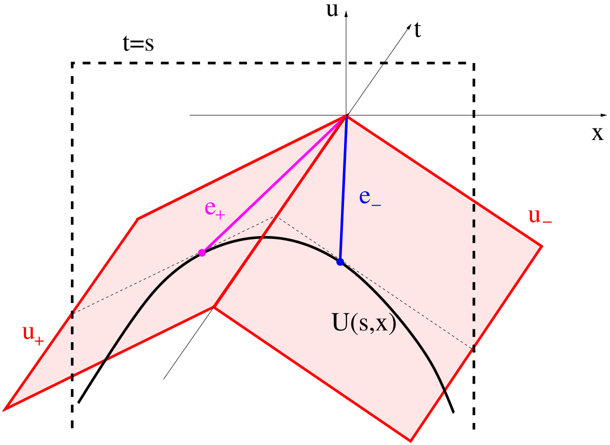


with

$$t < t' < 0$$

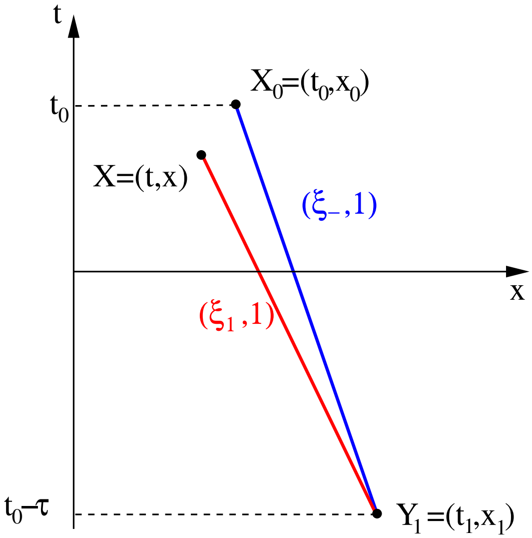
characteristic vectors

$$e_+ = (\xi_+, 1, \mathcal{L}(\xi_+)) \quad \text{and} \quad e_- = (\xi_-, 1, \mathcal{L}(\xi_-))$$



Step 4 : consequence of rigidity in ξ_-

If $u(X_0) < u_+(x_0)$, then $u(X_0) = u(Y_1) + \tau\mathcal{L}(\xi_-)$.



with $\tau >> 1$.

Step 4 : consequence of rigidity in ξ_-

For $X = (t, x)$, we have

$$\begin{array}{rll} & u(X) - u(X_0) & \\ \leq & -u(X_0) + u(Y_1) + (t - t_1)\mathcal{L}(\xi_1) & \text{with} \quad \xi_1 := \frac{x - y_1}{t - t_1} \\ = & (t - t_0)\mathcal{L}(\xi_1) + \tau \left\{\mathcal{L}(\xi_1) - \mathcal{L}(\xi_-)\right\} & \text{with} \quad t - t_1 = (t - t_0) + \tau \\ = & (t - t_0)\mathcal{L}(\xi_1) + \tau \int_0^1 d\sigma\ \bar{\xi} \cdot D\mathcal{L}(\xi_- + \sigma\bar{\xi}) & \end{array}$$

and

$$\bar{\xi} := \xi_1 - \xi_- = \tau^{-1} A + o(\tau^{-1}) \quad \text{with} \quad A := x - x_0 - (t - t_0)\xi_-$$

Step 4 : consequence of rigidity in ξ_-

As $\tau \to +\infty$, we get

$$u(X) - u(X_0) \leq (p_-) \cdot (x - x_0)$$

By symmetry $X \leftrightarrow X_0$, we get in $\{u < u_+\}$

$$u(t,x) - u(t_0,x_0) = (p_-) \cdot (x - x_0)$$

i.e.

$$u(t,x) = c + u_-(x) \quad \text{in} \quad \{u < u_+\}$$