De $\mathcal{P}_2(\mathbb{R}^d)$ à $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$: deux points de vue sur un problème de contrôle et son équation de HJB.

2 paires de lunettes...

Chloé Jimenez (Univ Brest)
Antonio Marigonda (Verona), Marc Quincampoix (Brest)

Plan

1. The Wasserstein Space
2. A multi-agent optimal control problem
3. Hamilton-Jacobi in the space of Wasserstein
4. Extending the Hamiltonian in $L^2_{\mathbb{P}}$ in a regular way

The Wasserstein space

$$\mathcal{P}_2(\mathbb{R}^d) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^2 \, d\mu(x) < +\infty \right\}$$

The Wasserstein Distance

For $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$:

$$\Pi(\mu,\nu) = \{\gamma \in \mathcal{P}_2(\mathbb{R}^{2d}) : \gamma \text{ has marginals } \mu \text{ and } \nu\}$$

$$W_2(\mu,\nu) = \min_{\gamma \in \Pi(\mu,\nu)} \left\{ \left(\int_{\mathbb{R}^{2d}} |y - x|^2 d\gamma(x,y) \right)^{1/2} \right\}$$

Probability space

- $(\Omega, \mathcal{B}(\Omega), \mathbb{P}) = ([0,1], \mathcal{B}([0,1]), \mathcal{L}^1 \llcorner_{[0,1]})$

Probability measures as laws

- For all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ it exists $X \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ such that:

$$\mu = X \sharp \mathbb{P}, \quad \mathbb{P}_X = \mu, \quad \text{the law of } X \text{ is } \mu$$

  the image measure of $\mathbb{P}$ by X is μ: $\mu(A) = \mathbb{P}(X^{-1}(A))$.
- For all $\gamma \in \Pi(\mu, \nu)$, it exists $X, Y \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ such that:

$$\gamma = (X, Y)\sharp \mathbb{P}, \quad X \sharp \mathbb{P} = \mu, \, Y \sharp \mathbb{P} = \nu.$$

Through Hilbertian glasses

Wasserstein space

$$W_2(\mu,\nu) = \min \left\{ \|X - Y\|_{L^2_{\mathbb{P}}} : \quad X\sharp\mathbb{P} = \mu,\, Y\sharp\mathbb{P} = \nu \right\}.$$

$\mathcal{P}_2(\mathbb{R}^d)$ as a quotient

- $X \sim X'$ iff $X\sharp\mathbb{P} = X'\sharp\mathbb{P}$
- $\mathcal{P}_2(\mathbb{R}^d) \equiv L^2_{\mathbb{P}}(\Omega)/\sim$

Transport maps

Transport plans supported on graphs

$$\gamma = (Id, T)\sharp\mu, \quad \mu \in \mathcal{P}_2(\mathbb{R}^d),\ T \in L^2_\mu(\mathbb{R}^d, \mathbb{R}^d)$$

$$\int \varphi(x,y)\, d\gamma(x,y) = \int \varphi(x, Tx)\, d\mu(x).$$

Transport plans supported on graphs

$$\gamma = (X, T \circ X)\sharp\mathbb{P}, \quad X \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d),\ T \in L^2_{X\sharp\mathbb{P}}(\mathbb{R}^d, \mathbb{R}^d),$$

$$\int \varphi(x,y)\, d\gamma(x,y) = \int \varphi(X, T \circ X)\, d\mathbb{P}.$$

Maps and lifts

Lifts and rearrangement invariance

- Let $u : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ we define the **lift** of u as:

$$U : X \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \mapsto u(X\sharp\mathbb{P}) \in \mathbb{R}.$$

- Then $U : L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \to \mathbb{R}$ is **rearrangement invariant**:

$$X\sharp\mathbb{P} = Y\sharp\mathbb{P} \Rightarrow U(X) = U(Y).$$

- u is continuous $/W_2$ **iff** it lift U is continuous $/\|\cdot\|_{L^2_{\mathbb{P}}}$.

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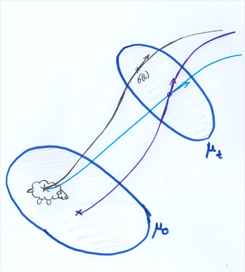
Controled trajectories: a toy example

A toy example: controlling the trajectory of a herd of sheeps

- $\eta \in \mathcal{P}(\mathcal{C}([t_0, T], \mathbb{R}^d))$
- $e_t(\sigma) = \sigma(t)$, $\mu_t := e_t \sharp \eta =$ "$\eta \circ e_t^{-1}$",
- η concentrated on curves σ with: $\dot{\sigma}(t) = f(\sigma(t), \underline{u}(t,\sigma), \mu_t)$ a.e. t
- f is regular, affine on $\underline{u}$
- $\eta = \eta^{t,x} \otimes \mu_t(x)$



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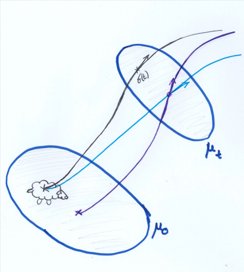

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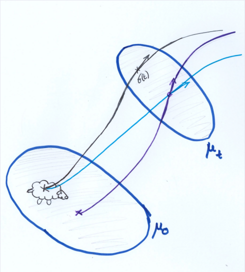


Controled trajectories: a toy example

Some computations with hands

- $\mu_t := e_t \sharp \eta, \quad \eta = \eta^{t,x} \otimes \mu_t(x)$
  Dynamic: $\dot{\sigma}(t) = f(\sigma(t), \underline{u}(t,\sigma), \mu_t)$
- We integrate the dynamic with respect to $\eta^{t,x}$:

$$v_t(x) := \int \dot{\sigma}(t)\, d\eta^{t,x}(\sigma) = \int f(\sigma(t), \underline{u}(t,\sigma), \mu_t)\, d\eta^{t,x}(\sigma)$$

$$= f\left(x, \int \underline{u}(t,\sigma) d\eta^{t,x}(\sigma), \mu_t\right) \ \mu_t\text{-a.e.}$$

Controled trajectories: a toy example

Dynamic in the space of Wassertein

$$v_t(x) = f(x, \underline{w}(t,x), \mu_t) \quad \mu_t\text{-a.e.}$$

Where $\underline{w}$ is a control.

Trajectories in $AC^2([t_0, T], \mathcal{P}_2(\mathbb{R}^d))$ (Ambrosio, Gigli, Savaré)

We assume $t \mapsto \mu_t \in \mathcal{P}_2(\mathbb{R}^d)$ is in $AC^2([t_0, T], \mathcal{P}_2(\mathbb{R}^d))$,

$$\int_0^T \int_{\mathbb{R}^d} |v_t(x)|^2 \, d\mu_t(x) \, dt < +\infty$$

$$\partial_t \mu_t + \mathrm{div}(v_t \mu_t) = 0 \text{ in } \mathbb{R}^d \times]t_0, T[$$

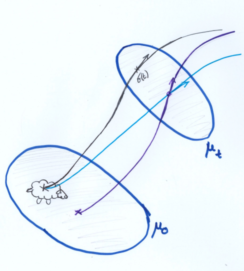
Be carefull

Remark

Define a plan γ_t:
$\int \varphi \, d\gamma_t(x,z) = \int \varphi(\sigma(t), \dot\sigma(t)) \, d\eta^{t,x}(\sigma)$

γ_t may not be of the type $(Id, v_t)\sharp\mu_t$

γ_t may not be supported on a graph!



A control problem

The Value function (Marigonda, Quincampoix, J. Marigonda, Quincampoix)

$$\mathcal{V}(t_0, \mu_0) := \inf_{(v_t, \mu_t), \underline{w}} \{\mathcal{G}(\mu_T) : \mu_{t_0} = \mu_0\}$$

admissible curves are in $AC^2([t_0, T], \mathcal{P}_2(\mathbb{R}^d))$ with

$$v_t(x) = f(x, \underline{w}(t, x), \mu_t) \quad \mu_t\text{-a.e.}$$

Assume for simplicity that $\mathcal{G}$ is Lipschitz so that $\mathcal{V}$ is regular.

How can we express this problem in $L^2_{\mathbb{P}}$?

Cavagnari, Lisini, Orrieri, Savaré,
J. Marigonda, Quincampoix and J.

How can we express this problem in $L^2_{\mathbb{P}}$?

A fonctionnal in $L^2_{\mathbb{P}}$ candidate to be the lift of $\mathcal{V}$

$$W(t_0, X) = \inf_{(X_t, \underline{u})} \left\{ \mathcal{G}(X_T \sharp \mathbb{P}) : \ X_{t_0} = X \right\}$$

admissible curves are in $AC^2([t_0, T], L^2_{\mathbb{P}}(\Omega)^d)$ with

$$\dot{X}_t(\omega) = f(X_t(\omega), \underline{u}(t, \omega), X_t \sharp \mathbb{P})$$

W is regular. Do we have that $W(\cdot, X) = \mathcal{V}(\cdot, \mu_0)$ if $X \sharp \mathbb{P} = \mu_0$?

Question

Given an admissible μ_t, can we find X_t admissible for W such that:

$$X_t \sharp \mathbb{P} = \mu_t?$$

From Wasserstein to $L^2_{\mathbb{P}}$

Building X_t

- The **Superposition Principle (AGS)** gives $\eta \in \mathcal{P}(\mathcal{C}([t_0, T], \mathbb{R}^d))$ associated to μ_t: $e_t \sharp \eta = \mu_t$
- It exists $T_\eta \in L^2_{\mathbb{P}}(\Omega, \mathcal{P}(\mathcal{C}([t_0, T], \mathbb{R}^d)))$ such that:

$$T_\eta \sharp \mathbb{P} = \eta.$$

- set $X_t := (e_t \circ T_\eta)$ so that:

$$X_t \sharp \mathbb{P} = (e_t \circ T_\eta) \sharp \mathbb{P} = e_t \sharp \eta = \mu_t$$

$$\dot{X}_t(\omega) = v_t(X_t(\omega)) = f(X_t(\omega), \underline{u}(t, \omega), X_t \sharp \mathbb{P})$$

with $\underline{u}(t, \omega) := \underline{w}(t, X_t(\omega))$.

From Wasserstein to $L^2_{\mathbb{P}}$

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Building X_t

- $\eta \in \mathcal{P}(\mathcal{C}([t_0, T], \mathbb{R}^d))$ associated to μ_t: $e_t \sharp \eta = \mu_t$
  Concentrated on curves such that $\dot{\sigma} = v_t(\sigma)$
- It exists $T_\eta \in L^2_{\mathbb{P}}(\Omega, \mathcal{P}(\mathcal{C}([t_0, T], \mathbb{R}^d)))$ such that:

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with $\underline{u}(t, \omega) := \underline{w}(t, X_t(\omega))$.

Value in $L^2_{\mathbb{P}}$

Consequence on the value

$$\mathcal{V}(t_0, \mu_0) \geq W(t_0, X_{t_0})$$

Problem

We cannot choose the starting point X_{t_0} among all Y_0 such that $Y_0 \sharp \mathbb{P} = \mu_0$

Solving problem

Important tool

Let $X, Y \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ such that $X\sharp\mathbb{P} = Y\sharp\mathbb{P}$.

Then, for any $n > 0$, there exists $\tau_n : \Omega \to \Omega$ bijective s.t.:

(i) $\|X \circ \tau_n - Y\|_{L^\infty_{\mathbb{P}}(\Omega,\mathbb{R}^d)} \leq \frac{1}{n}$,

(ii) $\tau_n\sharp\mathbb{P} = \tau_n^{-1}\sharp\mathbb{P} = \mathbb{P}$.

Of course $(X \circ \tau_n)\sharp\mathbb{P} = X\sharp\mathbb{P}$.

Solving the problem

Solving the problem: We cannot choose X_{t_0}

- Given Y_0 such that $Y_0 \sharp \mathbb{P} = \mu_0$ and X_t as above
- Using the tool, build $Y_t^n = X_t \circ \tau_n$ with:

$$(X_t \circ \tau_n) \sharp \mathbb{P} = \mu_t, \quad \| Y_0 - Y_{t_0}^n \| \leq \frac{1}{n}.$$

- $W(\cdot, Y_{t_0}^n) \leq \mathcal{V}(\cdot, \mu_0)$
- (!) the sequence of curves may not converge
- we don't care because of the regularity of W.

Consequence

$$\mathcal{V}(\cdot, \mu_0) \geq W(\cdot, Y_0) \quad \text{if } Y_0 \sharp \mathbb{P} = \mu_0.$$

From $L^2_{\mathbb{P}}$ to Wasserstein: The opposite inequality

From $L^2_{\mathbb{P}}$ to Wasserstein

- Let Y_t such that:

$$\dot{Y}_t(\omega) = f(Y_t(\omega), \underline{u}(t,\omega), Y_t\sharp\mathbb{P}).$$

- Set $\mu_t := Y_t\sharp\mathbb{P}$ and $\gamma_t = (Y_t, \dot{Y}_t)\sharp\mathbb{P}$
- $v_t(x) = \int y \, d\gamma_t^x(y)$ is the projection of $\dot{Y}_t$ on

$$H_{Y_t} = \{\varphi \circ Y_t : \ \varphi \in L^2_{Y_t\sharp\mathbb{P}}\}.$$

- Note that γ_t may devide masses.
- $t \mapsto \mu_t$ is admissible for $\mathcal{V}(Y_{t_0}\sharp\mathbb{P})$

From $L^2_{\mathbb{P}}$ to Wasserstein: The opposite inequality

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- Let Y_t such that:

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- $v_t(x) = \int y \, d\gamma_t^x(y)$, $v_t \circ X_t$ is the projection of $\dot{Y}_t$ on

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- Note that γ_t may devide masses.
- $t \mapsto \mu_t$ is admissible for $\mathcal{V}(Y_{t_0}\sharp\mathbb{P})$

Equality of problems in Wasserstein and $L^2_{\mathbb{P}}$

Equality of values: W is the lift of $\mathcal{V}$

$\mathcal{V}(t_0,\mu_0) = \inf_{\mu_{t_0}=\mu_0} \mathcal{G}(\mu_T)$	$V(t_0,X) = \inf_{X_{t_0}=X} \mathcal{G}(X_T\sharp\mathbb{P})$
$(\mu_s)_s \in AC^2([t_0,T],\mathcal{P}_2(\mathbb{R}^d))$ $v_t(x) = f(x,\underline{w}(t,x),\mu_t)$	$(X_t)_t \in AC^2([t_0,T],L^2_{\mathbb{P}}(\Omega)^d)$ $\dot{X}_t(\omega) = f(X_t,\underline{u}(t,\omega),X_t\sharp\mathbb{P})$

Remarks

The problem in $L^2_{\mathbb{P}}$ may have no solution, it depends on the choice of X.

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A result in Wasserstein space

Characterization of the value (Marigonda Quincampoix, J. Marigonda Quincampoix, J.)

The functionnal $\mathcal{V}$ is the unique **viscosity solution** of :

$$(HJ) \begin{cases} \partial_t u(t,\mu) + \mathcal{H}(\mu, D_\mu u(t,\mu)) = 0 & \forall (t,\mu) \in [0,T[\times \mathcal{P}_2(\mathbb{R}^d) \\ u(T,\mu) = \mathcal{G}(\mu) & \forall \mu \in \mathcal{P}_2(\mathbb{R}^d) \end{cases}$$

with $\mathcal{H}$ defined for $(\mu, p) \in \mathcal{P}_2 \times L^2_\mu$ as:

$$\mathcal{H}(\mu,p) := \inf_{\underline{u}} \left\{ \int_{\mathbb{R}^d} f(x, \underline{u}(x), \mu) \cdot p(x)\, d\mu(x) \right\}.$$

Fréchet Subdifferential in Wasserstein

Subdifferential (Gangbo, Nguyen and Tudorascu)

Let $(t_0, \mu_0) \in [0, T[\times \mathcal{P}_2(\mathbb{R}^d)$, we have $(p_t, p_\mu) \in D^- u(t_0, \mu_0)$ if:

- $p_\mu \in \mathcal{T}_{\mu_0}(\mathbb{R}^d)$
- for all (t, ν), $\gamma \in \Pi(\mu_0, \nu)$:

$$u(t, \nu) \geq u(t_0, \mu_0) + p_t(t - t_0) + \int p_\mu(x) \cdot (y - x)\, d\gamma(x, y)$$

$$+o\left(\sqrt{\|x - y\|^2_{L^2_\gamma} + |t - t_0|^2}\right).$$

Viscosity solution

Definition (Gangbo, Nguyen and Tudorascu)

- w is a **viscosity supersolution** of (HJ) if for all $(t_0, \mu_0) \in [0, T[\times \mathcal{P}_2(\mathbb{R}^d)$:

$$p_t + \mathcal{H}(\mu_0, p_\mu) \leq 0 \quad \forall (p_t, p_\mu) \in D^- w(t_0, \mu_0).$$

- define **subsolutions** in the same way.
- w is a **viscosity solution** if it is both a supersolution and a subsolution.

Translating the notion of viscosity solution in $L^2_{\mathbb{P}}$

Lift of $\mathcal{H}$

We set $H(X, p \circ X) := \mathcal{H}(X\sharp\mathbb{P}, p)$ for all $p \in L^2_{X\sharp\mathbb{P}}(\mathbb{R}^d, \mathbb{R}^d)$.

Fréchet sub-differential in $L^2_{\mathbb{P}}$

$(p_t, Z) \in D^- U(t_0, X)$ if for all (t, Y), it holds:

$$U(t, Y) \geq U(t_0, X) + p_t(t - t_0) + \langle Z, Y - X \rangle$$

$$+ o\left(\sqrt{\|Y - X\|^2_{L^2_{\mathbb{P}}} + |t - t_0|^2}\right)$$

Viscosity supersolution in Wasserstein space

$\mathcal{V}$ is a **supersolution** in the previous sense if

$$p_t + H(X, p_X \circ X) \leq 0 \quad \forall (p_t, p_X \circ X) \in D^- V(t, X)$$

with $p_X \in \mathcal{T}_{X \sharp \mathbb{P}}(\mathbb{R}^d)$.

Crucial points

- The lift of $\mathcal{H}$ is not defined on all $(L^2_{\mathbb{P}}(\mathbb{R}^d, \mathbb{R}^d))^2$,
- The definition of $\mathcal{P}_2$-viscosity super-solution involves only a part of the Fréchet subdifferential in $L^2_{\mathbb{P}}$
- U is a viscosity solution in $L^2_{\mathbb{P}} \Rightarrow u$ is a viscosity solution in Gangbo-Nguyen-Tudorascu sense.
- If we want V to be a viscosity solution in $L^2_{\mathbb{P}}$, we have **to extend** the lift H.

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Example

Lions' lecture at the Collége de France

- He considers $\mathcal{H}(\mu, p) = \|p\|^2_{L^2_\mu}$
- its Lift is $H(X, p \circ X) = \|p \circ X\|^2_{L^2_{\mathbb{P}}}$
- which he naturally extends as

$$\widetilde{H}(X, Y) = \|Y\|^2_{L^2_{\mathbb{P}}}$$

- this extension is regular.

We would like to do that in more general cases.

Literature

Literature

Several extensions already exist in the literature

- **Gangbo and Tudorascu**: $H_1(X, Z)$
- **Cavagnari, Marigonda, Quincampoix**: $H_2(X, Z)$.
- In both cases a projection of Z in H_X is used in order to turn (X, Z) into some $(X, p \circ X)$.
- In other terms they turn $\gamma(x, z) = (X, Z)\sharp\mathbb{P}$ into a plan supported on the graph of $x \mapsto \int z d\gamma^x$.
- They turn "**transport plans into transport plan supported on graphs**".

Properties of these extensions

Theorem (Gangbo, Tudorascu)

u is a **viscosity solution** in $\mathcal{P}_2(\mathbb{R}^d)$ in the sense of Gangbo, Nguyen, Tudorascu iff **its lift** U is a **viscosity solution** in $L^2_{\mathbb{P}}$ of the corresponding equation with H_1.

Problems

- Even if $\mathcal{H}$ is quite nice, $H_1(X, Z)$ and $H_2(X, Z)$ **are not continuous** on (X, Z).
- This comes from the fact that: **transport plans can be approximated by transport plans supported on graphs.**
- The lack of continuity prevents using classical results of Crandall and Lions.

Subdifferential in $[t_0, T[\times L^2_{\mathbb{P}}$

Fréchet Subdifferential

$(p_t, Z) \in D^- U(t_0, X)$ if $\forall (t, Y)$:

$$U(t, Y) \geq U(t_0, X) + p_t(t - t_0) + \langle Z, Y - X \rangle$$

$$+ o\left(\sqrt{\|Y - X\|^2_{L^2_{\mathbb{P}}} + |t - t_0|^2} \right).$$

Subdifferentials are plans (J. Marigonda, Quincampoix)

If U is **r.i.**, $(p_t, Z) \in D^- U(t_0, X)$ then:

for all (X', Z') with $(X', Z')\sharp\mathbb{P} = (X, Z)\sharp\mathbb{P}$: $(p_t, Z') \in D^- U(t_0, X')$.

Arguing with plans

$L^2_{\mathbb{P}}$-Subdifferential = AGS-subdifferential

- $\gamma = (X, Z)\sharp\mathbb{P}$
- $(p_t, Z) \in D^- U(t_0, X) \Leftrightarrow (p_t, \gamma) \in \partial^-_{AGS} u(t_0, \mu_0)$.

AGS-subdifferential

$(p_t, \gamma) \in \partial^-_{AGS} u(t_0, \mu_0)$ if:

- For all $\varpi(x, y, z) \in \mathcal{P}_2(\mathbb{R}^{3d})$ with $\pi_{x,z}\sharp\varpi = \gamma$, $\pi_y\sharp\varpi = \nu$:

$$u(t, \nu) \geq u(t_0, \mu_0) + p_t(t - t_0) + \int z \cdot (y - x)\, d\varpi(x, y)$$

$$+o\left(\sqrt{\|y - x\|^2_{L^2_\varpi} + |t - t_0|^2}\right).$$

New Hamiltonians

Extending Hamiltonians using plans (J.)

- Remember $\mathcal{H}(\mu_0, p)$ si defined for $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ and $p \in L^2_{\mu_0}(\mathbb{R}^d, \mathbb{R}^d)$
- Set $\tilde{\mathcal{H}}((Id, p)\sharp\mu_0) = \mathcal{H}(\mu_0, p)$.
- If $\tilde{\mathcal{H}}$ is uniformly continuous $/W_2$, we can extend it to all plans: $\tilde{\mathcal{H}}(\gamma)$.

A regular Hamiltonian on $L^2_{\mathbb{P}}$ (J.)

- By construction, $\tilde{\mathcal{H}}$ is continuous/W_2.
- By construction, its lift $\widetilde{H}$ is r.i. and continuous/L_2.
- It is in fact, the only regular extension of H.

Turning back to the example

$$\mathcal{H}(\mu, p) := \inf_{\underline{u}} \left\{ \int_{\mathbb{R}^d} f(x, \underline{u}(x), \mu) \cdot p(x)\, d\mu(x) \right\}$$

$$\widetilde{H}(X, Z) = \inf_{\underline{u}} \left\{ \int_{\Omega} f(X, \underline{u}(X, Z), X\sharp\mathbb{P}) \cdot Z\, d\mathbb{P} \right\}.$$

Turning back to the example (J.)

$$\widetilde{H}(X,Z) = \inf_{\underline{u}} \left\{ \int_\Omega f(X, \underline{u}(X,Z), X\sharp\mathbb{P}) \cdot Z \, d\mathbb{P} \right\}.$$

The function V is the unique solution in the usual $L^2_{\mathbb{P}}$-sense of:

$$\begin{cases} \partial_t U(t,X) + \widetilde{H}(X, D_X U(t,X)) = 0, \\ U(T,X) = \mathcal{G}(X\sharp\mathbb{P}). \end{cases}$$

Key point: the dynamic programming principle satisfied by $\mathcal{V}$ implies a dpp in $L^2_{\mathbb{P}}$ for V.

Other articles in the same spirit

- **Cavagnari, Savaré, Sodini,** *Dissipative probability vector fields and generation of evolution semigroups in Wasserstein spaces*, 2023
- **Bertucci,** *Stochastic optimal transport and Hamilton-Jacobi-Bellman equations on the set of probability measures*, preprint.

THE END

Thank you for your attention!