Dissipative node / interface coupling of scalar conservation laws

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based on joint works
with Karima Sbihi (2007, 2015), Clément Cancès (2013, 2015), Giuseppe Coclite and Carlotta Donadello (2017, 2024+ε)

Special thanks (inspiration):
Cyril Imbert & Régis Monneau (2014/17)

In memoriam : Serguei K. Godunov (1929 – 2023)

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Prerequisites / terminology :

- Scalar Conservation Law (SCL):

$$\partial_t u + \partial_x f(u) = 0 \quad + \text{ Initial Condition (IC)}$$

- [Kruzhkov'70] notion of entropy solution (x in the whole space) well-posedness for $L^1 \cap L^\infty$ IC, including the L^1 contraction

$$\|u(t,\cdot) - \hat{u}(t,\cdot)\|_{L^1_x} \leq \|u_0 - \hat{u}_0\|_{L^1_x}$$

  Entropy solution $\equiv$ limit of Vanishing Viscosity approximations.

- Riemann problems are Cauchy problems for pure-jump IC.
  They are building blocks for theory / for numerical schemes.
  Riemann solver = procedure or formula for solving Riemann pbs.

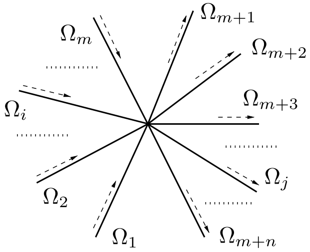
- [Godunov'59] Godunov flux, derived from the Riemann solver : an influential tool in Finite Difference / Finite Volume schemes

In God(unov) we trust

Overview: what's the story?

Network of roads. Well-posedness for wide families of node conditions?

Traffic on network:
m incoming roads
n outgoing roads
focus at a junction (node)



The problem structure:

- On each ray (road) : a classical, well-understood PDE, either SCL (Scalar conservation law) or HJ (Hamilton-Jacobi)
- At the node, a specific condition (node coupling / transmission)

Goals:

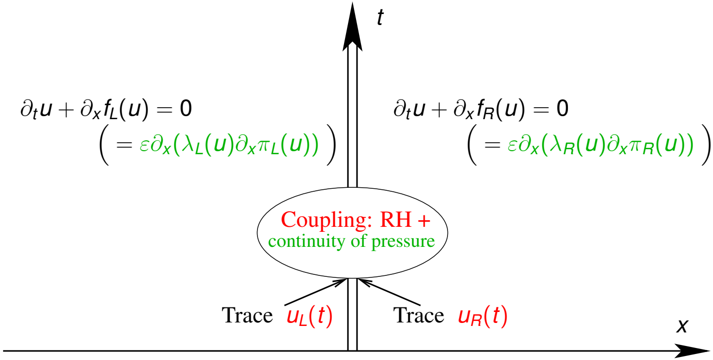
- address many node conditions within a common formalism
- benefit from abstract structures behind the problem
- relate/discriminate SCL and HJ -based models of network traffic cf. [Cardaliaguet-Forcadel-Girard-Monneau '24]

Example from porous media

Example: Buckley-Leverett equation as vanishing capillarity limit

Two-rock $1-1$ **junction:** Buckley-Leverett equation
in 1D medium made of two rocks with distinct physical properties

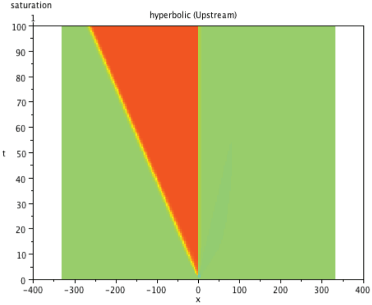
$$\partial_t u + \partial_x \Big(f_L(u) \mathbb{1}_{x<0} + f_R(u) \mathbb{1}_{x>0} \Big) = 0$$



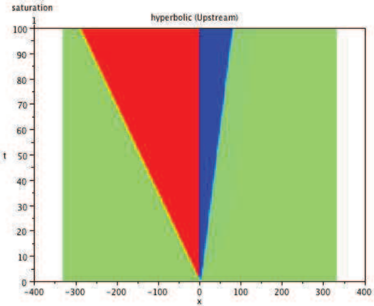
NB: the nonlinearities $\pi_{L,R}$ (capillary pressures) and $\lambda_{L,R}$
enter the model for $\varepsilon > 0$ but don't enter the limit model
$\Rightarrow$ how the Interface Coupling can keep memory of $\pi_{L,R}$, $\lambda_{L,R}$?

Different Interface Coupling Conditions lead to different solutions

$\varepsilon = 0$: Simulations for a constant initial condition and given f_L, f_R



(a) Numerical solution for constant datum



(b) Another numerical solution, same datum

Only difference between the two models:

different choice of capillary pressure profiles π_L, π_R

$\rightsquigarrow$ different interface (node) condition

$\rightsquigarrow$ different node Riemann solver

Can be seen as a class of models: common well-posedness theory.

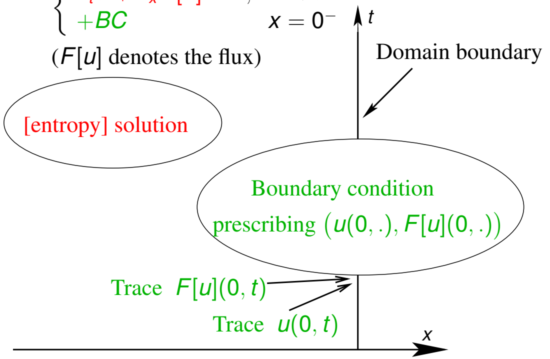
Laplacian : Classic BC & Monotonicity

Classic BC for evolution equations in divergence form (think Laplacian)

A starter: evolution PDE in divergence form on $1-0$ network
$\equiv$ classic Boundary-Value Problem paradigm

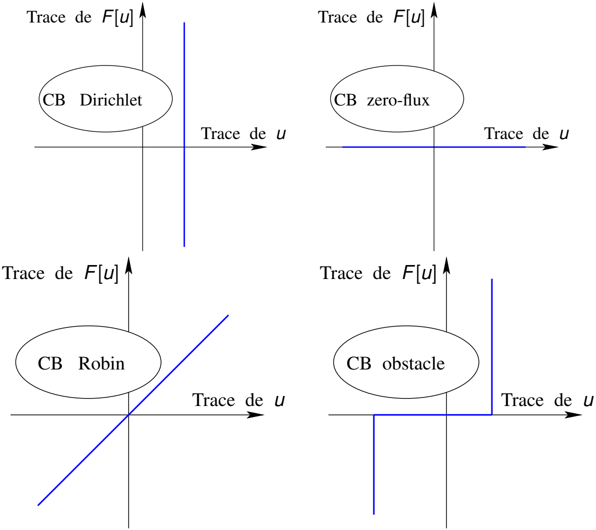
$$\begin{cases} \partial_t u + \partial_x F[u] = 0, & x<0 \\ +BC & x=0^- \end{cases}$$

($F[u]$ denotes the flux)



Think of $F[u] = -\nabla u$ (the standard Laplacian)...
...later, we'll rather think of SCL, with $F[u] = f(u)$!

Classic BC: monotonicity!



In all cases, $(u, F[u] \cdot n) \in \beta$ for some maximal monotone graph β

Monotonicity... monotonicities?

Think of the PDE $\partial_t u + \operatorname{div} F[u] = 0$ + the BC $(u, F[u]\cdot n) \in \beta$

- A graph $\beta \subset \mathbb{R}\times\mathbb{R}$ is monotone if

  for all pair $(u,F),(\hat u,\hat F) \in \beta$ any of the following holds
  - $(u-\hat u)(F-\hat F) \;\geq\; 0$
  - $\operatorname{sign}(u-\hat u)(F-\hat F) \;\geq\; 0$
  - . . . we'll see one more version later one

  Monotonicity $\rightsquigarrow$ stability and uniqueness of solutions:
  - in L^2, for the 1st version above
    (taking $(u-\hat u)$ for test function in the PDE)
  - in L^1, for the 2nd version
    (taking $\operatorname{sign}(u-\hat u)$ for test function in the PDE)
  - in L^p, for further versions of monotonicity and appropriate test fcts
- A monotone graph is maximal monotone
  if it admits no non-trivial monotone extension

  Maximality $\rightsquigarrow$ belief in / hope for solutions' existence

Scalar conservation law:

Bardos-LeRoux-Nédélec revisited

Dirichlet BC for conservation law $\rightsquigarrow$ Bardos-LeRoux-Nédélec

Dirichlet BC for the Laplacian:
While the trace of u is prescribed to a given value u^D,
the trace of $F[u] \cdot n = -\partial u / \partial n$ is free
$\rightsquigarrow$ wide enough choice for solutions' existence

Dirichlet BC for SCL:
When the trace of u is prescribed to a given value u^D,
the trace of $F[u] \cdot n = f(u) \cdot n$ is automatically prescribed
$\rightsquigarrow$ overdetermined problem, non-existence for most of data

BLN relaxation: [Bardos-LeRoux-Nédélec '79]
a rule, derived from analysis of Vanishing Viscosity approximation, prescribes a set $I(u^D)$ of values for the trace of u that is considerably larger than $\{u^D\}$
$\rightsquigarrow$ existence, uniqueness for the relaxed problem

Reinterpretation: [Dubois-LeFloch '89]
the BC graph β is projected on the graph of $f \cdot n$

Practical use, generalization: [A., Sbihi '06, '08, '15]
- The BLN relaxation /projection procedure can be described using the marvelous tool of Godunov function
- It can can be applied to any maximal monotone BC graph β

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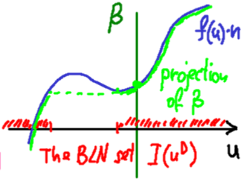
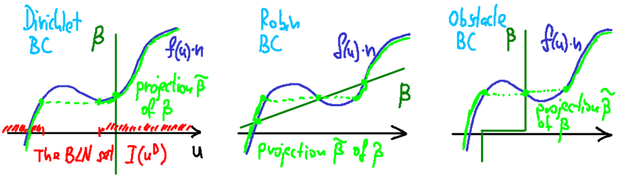


Visualization of BLN. Monotonicity. Godunov projection.



Structure of the projected graph: [A.-Sbihi'15]

$\tilde\beta$ is the closest to β maximal monotone subgraph of $f \cdot n$ (call it "canonical graph")

"Godunov representation" of $\tilde\beta$

The Godunov function can be used to encode the presence of boundary layer (passage from the true trace u to the "desired trace" $\tilde u$):

$$\mathrm{God}(a,b) = \begin{cases} \min_{[a,b]} f\cdot n & , \quad \text{if } a \le b \\ \max_{[b,a]} f\cdot n & , \quad \text{if } b \le a \end{cases}$$

$$\tilde\beta \;=\; \Big\{ (u,F) \,\Big|\, \exists(\tilde u, F) \in \beta \text{ s.t. } f(u)\cdot n = \mathrm{God}(u,\tilde u) = F \Big\}$$

Dicihlet case $\beta = \{u^D\} \times \mathbb{R}$:
the domain of $\tilde\beta$ is the Bardos-LeRoux-Nédélec set $I(u^D)$... call it "germ"!

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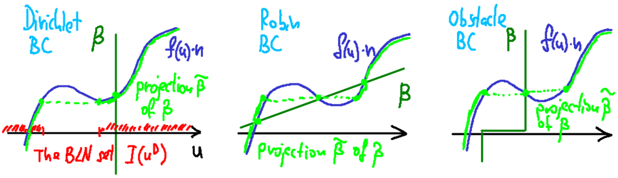

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Interface Coupling Conditions & Monotonicity

Examples for the Laplacian: Kirchhoff and Kedem-Katchalsky

Think of $\partial_t u + \operatorname{div} F[u] = 0$, $F[u] = -\nabla u$ (the Laplacian)
with inner interface $\Gamma = \{x_1 = 0\}$

Kirchhoff coupling:

$$\begin{cases} u|_{x_1=0^-} = u|_{x_1=0^+} & \text{continuity of } u \text{ on } \Gamma \\ F[u]\cdot n_-|_{x_1=0^-} + F[u]\cdot n_+|_{x_1=0^+} = 0 & \text{flux conservativity on } \Gamma \end{cases}$$

Well-known fact: Kirchhoff coupling $\iff$ the inner interface is "fake"

Kedem-K. coupling: [Kedem-Katchalsky '58],[Guarguaglini-Natalini]

$$\begin{cases} F[u]\cdot n_-|_{x_1=0^-} = C\big(u|_{x_1=0^-} - u|_{x_1=0^+}\big) & \text{a membrane condition on } \Gamma \\ F[u]\cdot n_+|_{x_1=0^+} = -C\big(u|_{x_1=0^-} - u|_{x_1=0^+}\big) & \text{(including flux conservativity)} \end{cases}$$

Condensed notation: one-sided traces $u_{L,R}$, $F_{L,R}$ fulfill

Kirchhoff

$$\begin{cases} u_L = u_R \\ F_L + F_R = 0 \end{cases}$$

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$$\begin{cases} F_L = C(u_L - u_R) \\ F_R = -C(u_L - u_R) \end{cases}$$

In both cases, solutions fulfill $((u_L, u_R), (F_L, F_R)) \in$ graph in $\mathbb{R}^2 \times \mathbb{R}^2$

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Kirchhoff / Transmission map / Flux limitation for SCL on 1—1 junction

Now, think of $\partial_t u + \partial_x F[u] = 0$, $F[u] = f(u)$ (the SCL)
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Condensed notation:

· one-sided (desired) traces $u_{L,R}$ of the solution
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porous medium applic. ("2-rocks")
$\pi_{L,R}$ capillary pressure profiles

Flux limitation ICC: [Colombo-Goatin'07],[A.'15], traffic applications

$$\begin{cases} u_L = u_R \\ F_L + F_R = 0, \; F_L \le F_{lim} \end{cases} \quad \text{OR} \quad \begin{cases} u_L > u_R \\ F_L = F_{lim} = -F_R \end{cases}$$

In all cases, what is called "solutions" in the above works fulfill
$\Big((u_L, u_R), (F_L, F_R)\Big) \in$ **"BLN-like" projected graph** in $\mathbb{R}^2 \times \mathbb{R}^2$

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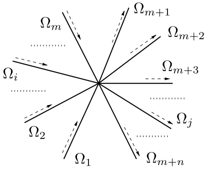
Framework of Node Coupling Conditions. Monotonicity...?

Network: incoming branches $\Omega_1, \ldots, \Omega_m$
outgoing branches $\Omega_{m+1}, \ldots, \Omega_{m+n}$
fluxes $F_\ell[\cdot]$ on Ω_ℓ, $\ell = 1, \ldots, m+n$



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· a (maximal monotone?) graph $\beta \subset \mathbb{R}^{m+n} \times \mathbb{R}^{m+n}$

Node Coupling encoded by $\left(\vec{u}\,,\vec{F}\right) \in \beta$

Monotonicity ? Depending on the uniqueness technique in use,

for all pair $(\vec{u}, \vec{F})\,,\,(\hat{\vec{u}}, \hat{\vec{F}}) \in \beta$ ask one of the following:

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- $\ldots\,\infty$-monotonicity ? a CoSSy possibility !

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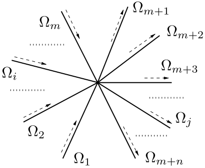

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"$L^1 D$ germs" theory for SCL recast into the ICC terminology

Interpretation of ICC vision in terms of [A.-Karlsen-Risebro'11]

1. BLN-like projection:

· The graph β is projected (description in terms of Godunov function)
· The BLN/Godunov projection $\tilde{\cdot} : \beta \to \tilde{\beta}$ preserves monotonicity/ies

2. Germ = Domain of the projected graph:

· The projected graph $\tilde{\beta}$ is fully determined by its domain
(we have $F_\ell = \pm F_\ell(u_\ell)$ with "$+$" on incoming, "$-$" on outgoing branches)

· call $Dom(\tilde{\beta})$ "germ", denote in $\mathcal{G}_\beta$

· 1-monotonicity of $\tilde{\beta} \iff L^1 D$ **property of the germ** $\mathcal{G}_\beta$

3. Maximality & Riemann problems:

· Maximality of the projected graph $\tilde{\beta}$ is inclear even if β is maximal

· The right property is **completeness of the germ** $\mathcal{G}_\beta$,
i.e., the ability to solve every Riemann problem at the node

Conclusion: Assume β is 1-monotone and defines a Riemann solver, $\mathcal{G}_\beta$ is **maximal** $L^1 D$ $\rightsquigarrow$ germs-based well-posedness theory applies

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Main objects: node Riemann solver / node Godunov flux / node germ.

1. Node Riemann problem: Given $\vec{r} = (r_1, \ldots, r_{m+n})$, find $(\vec{u}, \vec{F}) \in \beta$ s.t.

$$\begin{cases} \text{for } 1 \leq i \leq m & \text{God}_i(r_i, u_i) = F_i \\ \text{for } m+1 \leq i \leq m+n & \text{God}_j(u_j, r_j) = -F_j \end{cases}$$

· resolution is an intricate, β-dependent procedure!

· existence of a solution for all $\vec{r}$ means completeness for the germ

· monotonicity of β implies that the component $\vec{F}$ (fluxes) of the solution is uniquely defined (while $\vec{u}$ may be non-unique)

2. Node Godunov flux God_β:

If the above problem has a solution, this defines a map,

$$\text{God}_\beta : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}, \quad \vec{r} \mapsto \vec{F}$$

where $\vec{F}$ is the 2nd component of a solution $(\vec{u}, \vec{F}) \in \beta$ in 1..

3. Node germ G_β is the set of equilibria of the Riemann solver,

i.e., G_β is the set of all $\vec{r} \in \mathbb{R}^{m+n}$, which means that

$$\begin{cases} \text{for } 1 \leq i \leq m & f_i(r_i) = \text{God}_i(r_i, u_i) = F_i \\ \text{for } m+1 \leq i \leq m+n & f_j(r_j) = \text{God}_j(u_j, r_j) = -F_j \end{cases}$$

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Basics of the well-posedness theory with ICC β

Definition 1 of solution

A solution is a function defined on the network, being per-branch Kruzhkov entropy solution, and which traces at the node are in the germ $\mathcal{G}_\beta$.

Uniqueness for Definition 1: just like [A.-Karlsen-Risebro'11]

1-Monotonicity of β $\rightsquigarrow$ L^1-dissipativity of $\mathcal{G}_\beta$

$\rightsquigarrow$ interface terms reinforce the Kruzhkov contraction $\rightsquigarrow$ uniqueness

Definition 2 of solution, existence

A solution is a function defined on the network, satisfying adapted entropy inequalities (Kruzhkov's $k \in \mathbb{R}$ replaced by per-branch constants $\vec{k} \in \mathcal{G}_\beta$).

Existence for Definition 2: like [A.-Cancès'15],[A.-Coclite-Donadello'17]

· Definition of Godunov functions God_ℓ

$\rightsquigarrow$ existence of profiles (viscous, numerical...) with endpoints $\vec{k} \in \mathcal{G}_\beta$

· Contraction between approx. solutions & profiles

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Def. 2 $\Longrightarrow$ Def. 1 $\Longrightarrow$ Def. 2: like [A.-Karlsen-Risebro'11]

· Completeness of $\mathcal{G}_\beta$ $\rightsquigarrow$ maximality of $\mathcal{G}_\beta$ $\rightsquigarrow$ "'Def 2. $\Rightarrow$ Def 1."

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Examples: standard (Kirchhoff) VV on networks ; Transmission maps

Vanishing Viscosity on networks: [A.-Coclite-Donadello'17]
The ICC is mere Kirchhoff, given by

$$\beta = \left\{ (\vec{u}, \vec{F}) \,\middle|\, u_1 = \cdots = u_{m+n},\ \sum_{\ell=1}^{m+n} F_i = 0 \right\}$$

Solving Riemann problems: given $\vec{r}$, find a value $p \in \mathbb{R}$ s.t.

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a scalar monotone equation on $p \rightsquigarrow$ solution found with dichotomy method

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Given increasing capillary pressure profiles $\pi_{L,R}$, the ICC is given by

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Solving Riemann problems $\iff$ solving a scalar equation
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Formalism does not require conservativity, it can be applied e.g. to the Burgers-particle model of [Lagoutière-Seguin-Takahashi'07]

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A new example: Velocity Limitation in road traffic

Velocity limitation on $1-1$ junction: [A.-Rosini'25++?]
Question to [Colombo-Goatin'07] flux-limited model:

Why not velocity limitation ?

Formal velocity limitation ICC:

$$\beta = \Big\{(u,u,F,-F) \mid u \text{ arbitrary},\ F \le V_{lim}\, u\Big\} \text{ (classical Kirchhoff part)}$$
$$\bigcup \Big\{(u_L,u_R,F,-F) \mid u_L > u_R,\ F = V_{lim}\, u_L\Big\} \text{ (non-classical part)}$$

NB: This includes modeling assumptions (Rosini)

Calculations $\rightsquigarrow$ BLN-like projection $\tilde{\beta}$.

The projection turns out to be the same as for the flux limitation, at some level F_{lim} depending on V_{lim} and of f !

Conclusion:

By a BLN-like mechanism, velocity limitation amounts to a flux limitation
Conclusion supported by micro-macro (Follow-the-Leader) hydrodynamic limit numerics [A.-Rosini'19] and analysis [Storbugt'24]

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Kedem-Katchalsky node conditions on networks: first results.

The starting-point results

Following [Guarguaglini-Natalini] for Kedem-Katchalski coupling in parabolic case, [Coclite-Donadello'20] prove:

- Existence of KK-VV approximations (= KK coupling at the viscous level)
- Compactness of approximations as the viscosity parameter tends to 0^+
- L^1 contraction at the level of the viscous problem

$\rightsquigarrow$ the KK-VV limits form one (or many) L^1 contractive semigroups.

Question:

· Characterize the KK-VV limits intrinsically
· Prove uniqueness (intrinsic uniqueness / uniqueness of the KK-VV limit)

Failed attempts:

- · the language of connections [Adimurthi-Mishra-Gowda'05]
  · the language of flux limitation [Colombo-Goatin,...]
  seem inappropriate (cf. [Monneau, private comm.])
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Kedem-Katchalsky node conditions on networks: completing the study.

Summary of the result [A.-Coclite-Donadello'24+?]

ICC description of KK conditions ; BLN-kind Godunov projection framework
$\rightsquigarrow$ Node Riemann solver, Node Godunov flux, Node germ
$\rightsquigarrow$ intrinsic characterization of KK-VV limits & well-posedness

Key ingredient: abstract resolution of Riemann problems

· Given $\vec{r}$ a Riemann datum at the node, rewrite the system as

$$(\beta + \gamma_{\vec{r}})\, u \ni 0$$

$$\gamma_{\vec{r}} : u \mapsto \Big(+\mathrm{God}_1(r_1, u_1), \ldots \quad \ldots, -\mathrm{God}_{m+n}(u_{m+n}, r_{m+n}) \Big)$$

· Observe $\gamma_{\vec{r}}$ is a (completely) monotone and Lipschitz graph

· Using the theory of *m*-accretive operators [Bénilan, Crandall, Pazy], solve

$$(\delta \mathrm{Id} + \beta + \gamma_{\vec{r}})\, u^\delta \ni 0$$

· Pass to the limit $\delta \to 0^+$ in u^δ using uniform bounds (due to *T*-accretivity)

Result: Consider SCL on network with (formal) Node Coupling.
Coupling prescribed by a maximal 1-monotone[1] graph $\beta \subset \mathbb{R}^{m+n} \times \mathbb{R}^{m+n}$
$\rightsquigarrow$ the limit of β-VV approximations is the unique $\mathcal{G}_\beta$ solution
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Conclusion and open question

Conclusions, and a very CoSSy open question

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  · the L^1D germs' theory of discontinuous-flux conservation laws
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Open: Can the 1-monotonicity be replaced by a different monotonicity ?
Can the ∞-monotonicity structure be exploited ?

- Node Riemann solver well defined for ANY monotonicity notion
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- HJ framework requires scalar Node Hamiltonian (F, not $\vec{F}$)
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Merci !

Thank you

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