First order Mean Field Games on networks

Claudio Marchi

University of Padova

joint work with: **Y. Achdou**, **P. Mannucci** and **N. Tchou**

Outline

1. A brief introduction to first order MFG

2. MFG on networks: controls = velocity
   - MFG equilibrium
   - Lipschitz continuity of optimal trajectories
   - mild solution
   - Hamilton-Jacobi problem on the network

3. Other MFG problems on networks

First order Mean Field Games: Euclidean setting

The Mean Field Games model was proposed by Lasry-Lions in 2006 for describing interactions among a very large ("infinite") number of, indistinguishable and individually negligible, agents when individual actions are related to mass behaviour and vice versa.

Model example: $\mathbb{R}^d$

The generic player, starting at point $x \in \mathbb{R}^d$ at time t, chooses the best trajectory γ so to minimize the cost

$$\int_t^T \left[\frac{|\gamma'(s)|^2}{2} + \ell[m(s)](\gamma(s), s)\right] ds + G[m(T)](\gamma(T)),$$

where $m(s)$ is the distribution of the population at time s.

MFG system

$$\begin{cases} (HJ) & -u_t + \frac{1}{2}|\nabla u|^2 - \ell[m(t)](x,t) = 0 & (x,t) \in (0,T) \times \mathbb{R}^d \\ (C) & m_t - \mathrm{div}\,(m\nabla u) = 0 & (x,t) \in (0,T) \times \mathbb{R}^d \\ & u(T,x) = G[m(T)](x) & x \in \mathbb{R}^d \\ & m(0,x) = m_0(x) & x \in \mathbb{R}^d \end{cases}$$

where m_0 is the initial distribution of agents: $m_0 \geq 0$, $\int_{\mathbb{R}^d} m_0 dx = 1$.

- The first equation is a Hamilton-Jacobi equation for the value function u for a generic player.
- The second equation is a continuity equation for the density m of the population.

Theorem [PL Lions] (see also Cardaliaguet)

There exist $u \in W^{1,\infty}_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$ and $m \in C([0,T]; \mathcal{P}_1(\mathbb{R}^d))$, bounded, such that

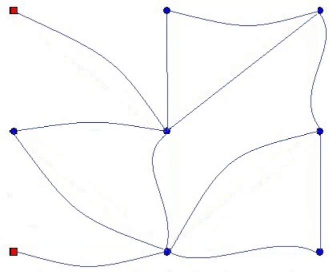
-) u solves (HJ) in the viscosity sense

-) m solves (C) in the sense of distributions.

AIM

Our aim is to study some classes of first order MFG when the dynamics of the agents take place in a network $\mathcal{G}$.

A network is a connected set $\mathcal{G}$ formed by a set of vertices $V := \{v_i\}_{i \in I_V}$ and a set of edges $E := \{J_i\}_{i \in I_J}$ connecting the vertices. We assume that $\mathcal{G}$ is embedded in $\mathbb{R}^d$ and that any two edges can intersect only at a vertex.



(a) An example of network

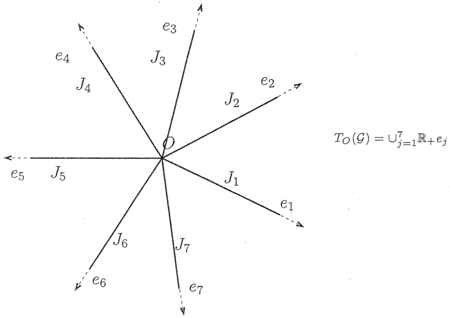
Literature for 1^{st} order MFG

- MFG on Euclidean spaces
  - *classical approach*
    - P.L. Lions' lectures at Collège de France 2012 - Cardaliaguet "Notes on Mean Field Games"
    - Cardaliaguet, DGA 2013
  - *Lagrangian approach for state constraints*
    - Benamou-Carlier, JOTA 2015
    - Cannarsa-Capuani, Springer-Indam, 2018
    - Cannarsa-Capuani-Cardaliaguet, ME 2019 & CVPDE 2021
    - Mazanti-Santambrogio, M^3AS 2019
- MFG on graphs (finite number of states)
  - Gomes-Mohr-Souza, JMPA 2010 & AMO 2013
- MFG on networks (2^{nd} order case)
  - Camilli-Marchi, SIAM JCO 2016
  - Achdou-Dao-Ley-Tchou, NHM 2019 & CVPDE 2020
- MFG on networks (1^{st} order MFG and Wardrop equilibrium)
  - Gomes *et al.* preprint

Control on the velocity – star shaped network

Star shaped network

For simplicity, consider a network $\mathcal{G}$ with N semi-infinite straight edges $(J_i)_{i=1,\dots,N}$ glued at the origin O. The edge J_i is the closed half-line $\mathbb{R}^+ e_i$. The vectors e_i are two by two distinct unit vectors.



Cost for a generic player

The state space is $\mathcal{G}$. The generic player aims at minimizing the cost

$$J(x,t,\gamma') \;=\; \int_t^T \left[\frac{|\gamma'(s)|^2}{2} + \ell[m(s)](\gamma(s),s)\right] ds + \; G[m(T)](\gamma(T))$$

where

$$\begin{aligned}
\ell[m](x,s) &= \sum_{i=1}^{N} \ell_i[m](x,s)\mathbf{1}_{x\in J_i\setminus\{O\}} + \ell_O[m](s)\mathbf{1}_{x=O} \\
G[m](x) &= \sum_{i=1}^{N} G_i[m](x)\mathbf{1}_{x\in J_i\setminus\{O\}} + G_O[m]\mathbf{1}_{x=O}
\end{aligned}$$

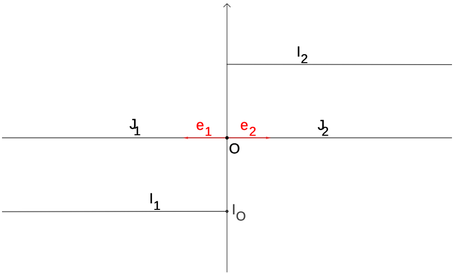
and

$$\begin{aligned}
\ell_O[m](s) &= \min\{\ell_*[m](s), \min_{i=1,\dots,N} \ell_i[m](O,s)\} \\
G_O[m] &= \min\{G_*[m], \min_{i=1,\dots,N} G_i[m](O)\}.
\end{aligned}$$

Main issues

- The running cost ℓ and the final cost G may be not continuous in O.
- The distribution of players may develop a singularity for any $t > 0$. This singularity may move in the network.

Example. Consider $\mathcal{G} = J_1 \cup J_2$, $m_0 = 1$ on $[0,1]J_2$, $G = 0$ and ℓ is



(c) A development of singularity

Consequence

We follow a Lagrangian approach.

Notations.

- $\Gamma = \{\gamma \in W^{1,2}(0,T;\mathbb{R}^d) : \gamma(\cdot) \in \mathcal{G}\}$, $\Gamma[x] = \{\gamma \in \Gamma : \ \gamma(0) = x\}$
- $\mathcal{P}(\Gamma) = \{\text{Borel probability measures on } \Gamma\}$
- $\forall t \in [0,T]$, the evaluation map is $e_t : \Gamma \to \mathcal{G}$ with $e_t(\gamma) = \gamma(t)$
- $\mathcal{P}_{m_0}(\Gamma) = \{\eta \in \mathcal{P}(\Gamma) : \ e_0 \# \eta = m_0\}$
- to each $\eta \in \mathcal{P}_{m_0}(\Gamma)$, we associate the cost

$$J^\eta(t,x,\gamma') = \int_t^T \left[\frac{|\gamma'(s)|^2}{2} + \ell[e_s\#\eta](\gamma(s),s)\right] ds + G[e_T\#\eta](\gamma(T))$$

and the corresponding set of optimal trajectories

$$\Gamma^\eta[x] = \left\{\gamma \in \Gamma[x] : \ J^\eta(0,x,\gamma') \le J^\eta(0,x,\tilde{\gamma}') \quad \forall \tilde{\gamma} \in \Gamma[x]\right\}.$$

Lemma (Existence of optimal trajectories)

For any $\eta \in \mathcal{P}_{m_0}(\Gamma)$ and $x \in \mathcal{G}$, $\exists$ an optimal trajectory starting at x.

Lemma (Approximation of admissible trajectories)

Let $x_n \to x$ and $\gamma \in \Gamma[x]$. Then, $\exists \gamma_n \in \Gamma[x_n]$ such that

$$\gamma_n \to \gamma \text{ unif.}, \qquad \gamma_n' \to \gamma' \text{ in } L^2, \qquad \lim_{n\to\infty} J^\eta(0,x_n,\gamma_n') = J^\eta(0,x,\gamma').$$

Definition

A measure $\eta \in \mathcal{P}_{m_0}(\Gamma)$ is a MFG equilibrium for m_0 if

$$\mathrm{supp}(\eta) \subset \bigcup_{x \in \mathrm{supp}(m_0)} \Gamma^\eta[x].$$

(Recall: $\Gamma^\eta[x] = \{\gamma \in \Gamma[x] : \; J^\eta(0, x, \gamma') \leq J^\eta(0, x, \tilde{\gamma}') \quad \forall \tilde{\gamma} \in \Gamma[x]\}$.)

Theorem 1 (Existence of a MFG equilibrium)

Assume

- $m_0 \in \mathcal{P}(\mathcal{G})$ has compact support
- $\ell_i[\cdot] : \mathcal{P}(\mathcal{G}) \to C^0(\mathcal{G} \times [0, T])$ are bounded and continuous
- $G_i[\cdot] : \mathcal{P}(\mathcal{G}) \to C^0(\mathcal{G})$ are bounded and continuous

for $i = *, 1, \ldots, N$.
Then, there exists a MFG equilibrium η associated with m_0.

Proof (sketch)

Following the *Lagrangian approach* of Cannarsa-Capuani, we introduce the *multivalued* map: for a suitable compact subset $\mathcal{K}$ of $\mathcal{P}_{m_0}(\Gamma)$,

$$E : \mathcal{K} \rightrightarrows \mathcal{K}$$

$$E(\eta) = \left\{ \hat{\eta} \in \mathcal{P}_{m_0}(\Gamma) : \quad \mathrm{supp}(\hat{\eta}) \subset \bigcup_{x \in \mathrm{supp}(m_0)} \Gamma^{\eta}[x] \right\}$$

and we apply Kakutani fixed point Theorem to obtain a MFG equilibrium. To this end, we have to prove

a) $\forall \eta \in \mathcal{K}$, $E(\eta)$ is a nonempty set

b) $\forall \eta \in \mathcal{K}$, $E(\eta)$ is a convex subset of $\mathcal{K}$

c) the map E fulfills the closed graph property.

For C sufficiently large, we consider the compact set

$$\mathcal{K} = \{\eta \in \mathcal{P}_{m_0}(\Gamma) : \mathrm{supp}(\eta) \subset \{\gamma : \|\gamma'\|_2 \le C, \|\mathrm{dist}(\gamma(\cdot), O)\|_\infty \le C\}\}.$$

Step 1. $\forall \eta \in \mathcal{K}$, $E(\eta)$ is a nonempty convex set.
Step 2. E fulfills the closed graph property:

$$\text{if } \eta^n \in \mathcal{K} \text{ with } \eta^n \to \eta \text{ and } \hat{\eta}^n \in E(\eta^n) \text{ with } \hat{\eta}^n \to \hat{\eta}, \text{ then } \hat{\eta} \in E(\eta).$$

- Disintegration theorem for $\hat{\eta}$:

$$\int_\Gamma f(\gamma)\hat{\eta}(d\gamma) = \int_{\mathcal{G}} \Big(\int_{\Gamma[x]} f(\gamma)\hat{\eta}_x(d\gamma)\Big) m_0(dx).$$

- Kuratowski convergence theorem: $\forall \gamma \in \mathrm{supp}\hat{\eta}_x$, $\exists \{\gamma_n\}_n$ with $\gamma_n \in \mathrm{supp}\hat{\eta}^n$ and $\gamma_n \to \gamma$ unif.. Hence: $\gamma_n \in \Gamma^{\eta^n}[\gamma_n(0)]$ and $\gamma_n(0) \to \gamma(0) = x$.
- The multivalued map $(x, \eta) \mapsto \Gamma^\eta[x]$ has the closed graph property: $\gamma \in \Gamma^\eta[x]$.

Theorem 2 (Lipschitz continuity of optimal trajectories)

Assume $\ell_i[m](\cdot,t),\ G_i[m](\cdot) \in C^2(J_i)$ with

$$\|\ell_i[m](\cdot,t)\|_{C^2(J_i)},\ \|G_i[m](\cdot)\|_{C^2(J_i)} \leq K \quad \forall t \in [0,T], m \in \mathcal{P}(\mathcal{G}).$$

Then, for any MFG equilibrium η, there holds

$$\|\gamma'\|_\infty \leq V \qquad \forall \gamma \in \Gamma^\eta[x]$$

where V is a constant depending only on $\mathcal{G}$, K and $|x - O|$.

Proof (sketch)

Consider $\gamma \in \Gamma^\eta[x]$.

- Case 1. γ is inside an edge: Euler-Lagrange condition. If $\gamma \in J_i \setminus \{O\}$ in (t_1, t_2), then $\gamma''(t) = \partial_x \ell_i[e_t \# \eta](\gamma(t), t)$ in (t_1, t_2).
- Case 2. γ ends inside an edge: trasversality condition. If $\gamma(T) \in J_i \setminus \{O\}$, then $\gamma'(T) = -\partial_x G_i[e_T \# \eta](\gamma(T))$.
- Case 3. γ occupies twice O. If $\gamma(t_1) = \gamma(t_2) = O$ with $t_1 \neq t_2$, and $\gamma(t) \in J_i \setminus \{O\}$ for $t \in (t_1, t_2)$ then $\exists t_* \in (t_1, t_2)$ s.t. $\gamma'(t_*) = 0$.
- Case 4. γ starts inside an edge. If $x \in J_i \setminus \{O\}$, then $|\gamma'(0)| \leq C$ for a constant C depending only on $|x - O|$.

Definition

A couple (u, m) is a mild solution to the MFG if there exists a MFG equilibrium $\eta \in \mathcal{P}_{m_0}(\Gamma)$ such that

- $m(t) = e_t \# \eta \qquad \forall t \in [0, T]$
- u is the value function associated to η:

$$u(t,x) = \inf_{\gamma \text{ adm.}} J^\eta(t,x,\gamma').$$

Corollary (Existence and regularity of a mild solution)

Under the assumptions of Theorem 1

(*i*) There exists a mild solution (u, m) to the MFG.

Moreover, under the assumptions of Theorem 2

(*ii*) m belongs to $Lip(0, T; \mathcal{P}(\mathcal{G}))$

(*iii*) u is locally Lipschitz continuous in $\mathcal{G} \times (0, T)$.

Proof (sketch)

(*i*) is an immediate consequence of Theorem 1.

(*ii*) is obtained following the same arguments of Theorem 1 and taking advantage of Theorem 2.

(*iii*) (*a*) Lipschitz continuity in space (using competitor)
(*b*) local Lipschitz continuity in time (using (*a*) and Theorem 2).

Hamilton-Jacobi problem on the network

Question

Does u solve a Hamilton-Jacobi problem defined on $\mathcal{G}$?

Hamilton-Jacobi on networks: [Achdou-Camilli-Cutrì-Tchou'13], [Imbert-Monneau-Zidani'13], [Camilli-Schieborn'13], [Imbert-Monneau'17], [Barles-Briani-Chasseigne'14], [Barles-Chasseigne'15], [Achdou-Oudet-Tchou'15], [Lions-Souganidis'16], [Graber-Hermosilla-Zidani'17], [Morfe'20], [Carlini-Festa-Forcadel'20], [Fayad-Forcadel-Ibrahim'22], [Barles-Chasseigne, ppt], [Siconolfi'22]....

Answer

YES.

We first need to introduce the HJ operators.

Notations: test functions and HJ operators

- $C^1(\mathcal{G}) = \{\varphi \in C(\mathcal{G});\ \varphi_{|J_i} \in C^1(\mathbb{R}^+ e_i), \quad \forall i = 1, \dots, N\}$
- for $\varphi \in C^1(\mathcal{G})$,

$$D\varphi(x) = \begin{cases} D\varphi_{|J_i} & \text{if } x \in J_i \setminus \{O\} \\ \left(D\varphi_{|J_1}, \dots, D\varphi_{|J_N}\right) & \text{if } x = O; \end{cases}$$

  and analogously for $\varphi \in C^1(\mathcal{G} \times [0, T])$.
- Hamilton-Jacobi operators. Fix a MFG equilibrium $\eta \in \mathcal{P}_{m_0}(\Gamma)$.
  $\forall x \in J_i, p \in \mathbb{R}$

$$\begin{aligned} H_i^\eta(x, t, p) &= \frac{|p|^2}{2} - \ell_i[e_t \# \eta](x, t) \\ H_{i,+}^\eta(O, t, p) &= \begin{cases} \frac{|\bar{p}|^2}{2} - \ell_i[e_t \# \eta](O, t) & \text{if } p \leq 0 \\ -\ell_i[e_t \# \eta](O, t) & \text{if } p > 0. \end{cases} \end{aligned}$$

Hamilton-Jacobi problem associated to η

$$(HJ_\eta) \qquad \begin{cases} -\partial_t u + H_i^\eta(x,t,Du) = 0 & \text{if } x \in J_i \setminus \{O\} \\ -\partial_t u + H_O^\eta(t,Du) = 0 & \text{if } x = O \\ u(T,x) = G[e_T \# \eta](x) & \text{on } \mathcal{G} \end{cases}$$

$$H_O^\eta(t,p) = \max\left\{-\ell_O[e_t\#\eta](t), \max_{i=1,\dots,N}\left\{H_{i,+}^\eta(O,t,p_i)\right\}\right\}$$

Definition: solutions

u is a subsolution (resp., a supersolution) if: for all $\varphi \in C^1((0,T)\times\mathcal{G})$ s.t. $u-\varphi$ has a maximum (resp., a minimum) at (t,x) there holds

$$\begin{array}{rcll} -\varphi_t(t,x) + H_i^\eta(x,t,D\varphi) & \leq & 0 & (\text{resp., } \geq 0) \quad \text{if } x \in J_i \setminus \{O\} \\ -\varphi_t(t,x) + H_O^\eta(t,D\varphi) & \leq & 0 & (\text{resp., } \geq 0) \quad \text{if } x = O. \end{array}$$

Proposition

The value function u is a solution to (HJ_η).

Work in progress: control on the acceleration

In a network, the generic player controls its acceleration $x''(s)$ with or without bounds on x''.

Main features:

- the state is (x, x')
- lack of local controllability
- viability set.

Thank You!