Inverse Design for scalar conservation laws

Vincent Perrollaz

Institut Denis Poisson, Université de Tours

Joint works with Rinaldo Colombo and Abraham Sylla.

Journées de lancement de l'ANR COSS

Outline of the talk

1. Conservation laws: Origin and Cauchy Problem
   - Origins
   - Regular solutions
   - Entropy solutions
2. Inverse Design for Homogeneous Laws: going back in time
3. The Non-Homogeneous Case

1. Conservation laws: Origin and Cauchy Problem
   - Origins
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2. Inverse Design for Homogeneous Laws: going back in time

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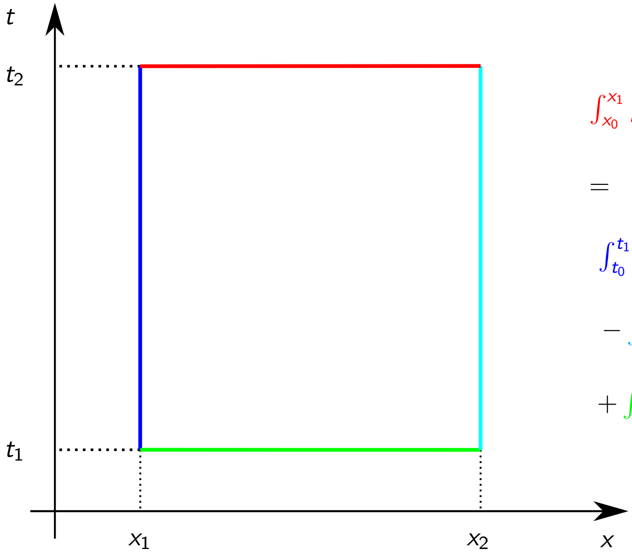
Integral Form



$$\int_{x_0}^{x_1} \rho(t_2, x)dx$$

$$=$$

$$\int_{t_0}^{t_1} F(t, x_1)dx$$

$$-\int_{t_0}^{t_1} F(t, x_2)dx$$

$$+\int_{x_0}^{x_1} \rho(t_1, x)dx$$

Differential Form

- Integral Form: ρ and F just L^1_{loc}.
- Differential Form:

$$\begin{aligned}
&\rho, F \in \mathcal{C}^1 \\
&\qquad \Longrightarrow \int_{x_1}^{x_2}\int_{t_1}^{t_2} \partial_t \rho(t,x)dtdx = -\int_{t_1}^{t_2}\int_{x_1}^{x_2} \partial_x F(t,x)dtdx \\
&\qquad \Longrightarrow \frac{\partial \rho}{\partial t}(t,x) + \frac{\partial F}{\partial x}(t,x) = 0.
\end{aligned}$$

Closure

- $F(t,x) = -\kappa \partial_x \rho(t,x)$ Heat Equation.

$$\partial_t \rho - \kappa \partial^2_{xx} \rho = 0.$$

- $F(t,x) = \frac{\rho^2(t,x)}{2}$ Burgers' equation (inspired by Euler's equation)

$$\partial_t \rho + \partial_x \left(\frac{\rho^2}{2} \right) = 0$$

- $F(t,x) = \rho(t,x) v_{\max} \left(1 - \frac{\rho(t,x)}{\rho_{\max}}\right)$ LWR equation

$$\partial_t \rho + \partial_x \left(\rho(t,x) v_{\max} \left(1 - \frac{\rho(t,x)}{\rho_{\max}} \right) \right) = 0$$

Characteristics' method

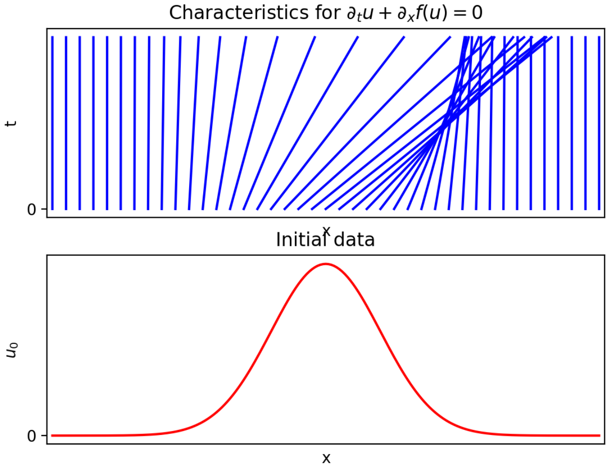
- $\begin{cases} \partial_t \rho + \partial_x f(\rho) = 0 \\ \rho(0,x) = \rho_0(x) \\ \rho \in \mathcal{C}^1 \end{cases}$
- $\partial_t \rho + \partial_x f(\rho) = 0 \qquad \Longrightarrow \partial_t \rho + f'(\rho)\partial_x \rho = 0,$
- $\begin{cases} q \in \mathcal{C}^1(\mathbb{R}^+;\mathbb{R}) \\ p(t) := \rho(t, q(t)) \end{cases} \qquad \Longrightarrow \dot{p}(t) = \partial_t \rho + \dot{q}(t)\partial_x \rho,$

$$" \Longrightarrow " \begin{cases} \dot{q}(t) = f'(p(t)) \\ \dot{p}(t) = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} p(t) = p(0) \\ \dot{q}(t) = f'(p(0)) \end{cases}$$

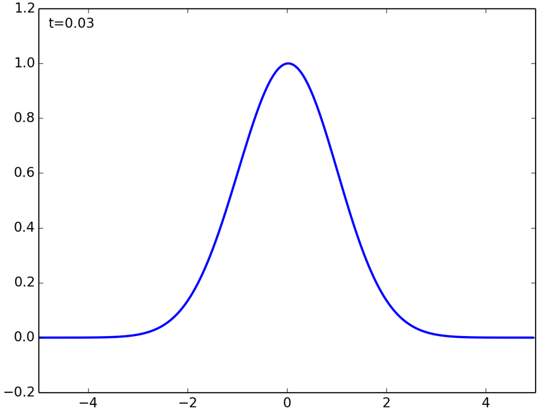
$$\Longrightarrow \rho(t,x) = \rho_0(x - tf'(\rho(t,x)))$$

In a picture with Burgers $f'(q) = q$



Animated

- Let the plane evolve according to $\dot{p} = 0$ and $\dot{q} = f'(p)$.
- Look at the evolution of the graph of $p = \rho(t, q)$.



Generic Blowup

$$\begin{cases} \partial_t \rho + \partial_x f(\rho) = 0, & t > 0, \quad x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x). \end{cases} \tag{1}$$

Theorem

For $f(\rho) = \frac{\rho^2}{2}$ (in fact convex or concave) and any $\rho_0 \in \mathcal{C}_c^\infty(\mathbb{R})$

$$\rho_0 \not\equiv 0 \Rightarrow \exists T > 0, \quad \exists X \in \mathbb{R}, \qquad \partial_x \rho(t,X) \underset{t \to T^-}{\to} -\infty.$$

But

$$||\rho(t,\cdot)||_{L^\infty(\mathbb{R})} \leq ||\rho_0||_{L^\infty(\mathbb{R})}.$$

Weak/Integral solutions

Three formulations, different regularity.

- Differential: $\partial_t \rho(t,x) + \partial_x f(\rho(t,x)) = 0,$
  $$\forall t > 0, \forall x \in \mathbb{R}$$
- Integral: $\dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_a^b \rho(t,x)dx = f(\rho(t,a)) - f(\rho(t,b))$
  $$\forall t > 0, \quad \forall a < b$$
- Weak: $\displaystyle\int_0^{+\infty}\int_{-\infty}^{+\infty} \rho(t,x)\partial_t\phi(t,x) + f(\rho(t,x))\partial_x\phi(t,x)dxdt = 0$
  $$\forall \phi \in \mathcal{C}_c^\infty((0,+\infty)\times\mathbb{R}).$$

Vanishing Viscosity and entropy solution.

- Analogy with gas dynamics

$$\partial_t \rho^\epsilon + \partial_x f(\rho^\epsilon) = \epsilon \partial_{xx}^2 \rho^\epsilon. \tag{2}$$

- For E convex, $Q' = E' f'$.

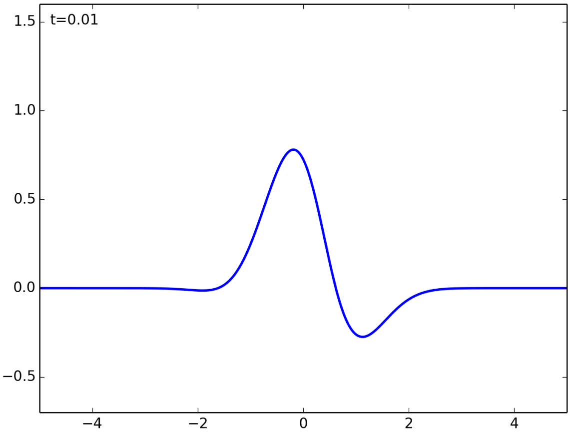
$$\partial_t E(\rho) + \partial_x Q(\rho) = \epsilon \partial_{xx}^2 E(\rho) - E''(\rho)(\partial_x \rho)^2.$$

- $\epsilon \to 0^+$ (**FORMALLY**):

$$\partial_t E(\rho) + \partial_x Q(\rho) \leq 0, \quad \mathcal{D}((0,+\infty) \times \mathbb{R})$$

- $\rho = (t,x) \mapsto (S_t \rho_0)(x)$, $(S_t)_{t \geq 0}$ semigroup of entropy solutions.

Simulations



What and Why!

Questions:

1. What is $S_t(L^\infty(\mathbb{R}))$?
2. Characterize $I_T(\omega) := \{\rho_0 \in L^\infty(\mathbb{R}) \, : \, S_T\rho_0 = \omega\}$ for $\omega \in L^\infty(\mathbb{R})$?

Reasons:

1. Irreversible dynamics for entropy solutions.
2. Entropy semigroup compactifying.
3. Sonic boom minimization. (Gosse-Zuazua 17)
4. Accident localization through tollgate estimates.
5. Control theory through Russell's extension method (Ancona-Marson 98, Horsin 98, ...).

Characterization of reachable states

Going back to Oleinik 56!

Definition

$$T > 0, \quad \omega \in L^\infty(\mathbb{R}) \qquad r_\omega^T(x) := x - Tf'(\omega(x)).$$

Theorem

For f convex,

$$I_T(\omega) \neq \emptyset \iff \omega \in S_T(L^\infty(\mathbb{R})) \iff r_\omega^T \text{ nondecreasing a.e.}$$

1. $\omega \in S_T(L^\infty(\mathbb{R})) \implies \omega \in \mathrm{BV}(\mathbb{R})$,
2. Not better: take r_ω^T Cantor's staircase.

Characterization of initial data

Theorem (Colombo-P. 2020)

$I_T(\omega) \neq \emptyset \implies$

1. $I_T(\omega)$ *convex,*
2. $I_T(\omega)$ *is a cone of vertex* ρ_0^**,*
3. $I_T(\omega)$ *is a* F_σ *set for the* L^1_{loc} *topology.*

Furthermore

1. $I_T(\omega)$ *singleton iff additionally* $\omega \in \mathcal{C}^0$
2. *otherwise unbounded* L^∞*, but locally* L^∞ *closed* L^1_{loc}*,*
3. *and there is no extremal facet of finite dimension besides* ρ_0^*

In fact complete characterization of $I_T(\omega)$.

ρ_0^*: reversing space and time

1. ω, T such that r_ω^T increasing
2. χ solution of

$$\begin{cases} \partial_s \chi + \partial_y f(\chi) = 0, \\ \chi(0, y) = \omega(-y) \end{cases}$$

3. χ isentropic: $\partial_s E(\chi) + \partial_y Q(\chi) = 0$ (in fact "regular")
4. $\rho(t, x) := \chi(T - t, -x)$ entropy solution

$$\partial_t \rho + \partial_x f(\rho) = 0.$$

5. $\rho_0^* := \rho(0) \in I_T(\omega)$

Tool: the Hopf-Lax formula

$$\begin{cases} \partial_t \rho + \partial_x (f(\rho)) = 0 \\ \rho(0) = \rho_0 \end{cases}$$

f strongly convex $\Updownarrow$ f^* Legendre transform

$$\begin{cases} r(t,x) := \underset{y\in\mathbb{R}}{\arg\min} \left(t f^* \left(\frac{x-y}{t}\right) + \int_0^y \rho_0(z)dz \right) \\ \rho(t,x) = f'\left(\frac{x-r(t,x)}{t}\right) \end{cases}$$

AND

enough minimizers depending only on ω!

Why?

1. LWR: different speed limit, different number of lanes:

$$\partial_t \rho(t,x) + \partial_x \left(\rho(t,x) v_{\max}(x)(1 - \frac{\rho(t,x)}{\rho_{\max}(x)}) \right) = 0.$$

2. Nonhomogeneous conservation laws:
   1. Richer geometry!
   2. ⚠ Litterature ⚠
3. Homogeneity in space too special:
   1. Vanishing viscosity for LWR?
   2. Infinitely many contracting semigroups of weak solutions!

Characteristics equation

- $$\begin{cases} \partial_t \rho + \partial_x \left(H(x,\rho)\right) = 0 \\ \rho(0,x) = \rho_0(x) \\ \rho \in \mathcal{C}^1 \end{cases}$$
- $\partial_t \rho + \partial_x \left(H(x,\rho)\right) = 0 \implies \partial_t \rho + \partial_2 H(x,\rho) \partial_x \rho = -\partial_1 H(x,\rho),$
- $\begin{cases} q \in \mathcal{C}^1(\mathbb{R}^+;\mathbb{R}) \\ p(t) := \rho(t,q(t)) \end{cases} \implies \dot{p}(t) = \partial_t \rho + \dot{q}(t) \partial_x \rho,$

$$\text{``} \implies \text{''} \begin{cases} \dot{q}(t) = \partial_2 H(q(t),p(t)) \\ \dot{p}(t) = -\partial_1 H(q(t),p(t)) \end{cases}$$

Inverse design results: The Same?

$$T > 0,\ \omega \in L^\infty(\mathbb{R}), \quad r_\omega^T(x) := q(0) \text{ where } \begin{cases} \dot q(t) = \partial_2 H(q(t), p(t)) \\ \dot p(t) = -\partial_1 H(q(t), p(t)) \\ q(T) = x, \quad p(T) = w(x). \end{cases}$$

Theorem (Colombo-P.-Sylla)

When $\forall x, \quad q \mapsto H(x, q)$ *strongly convex*

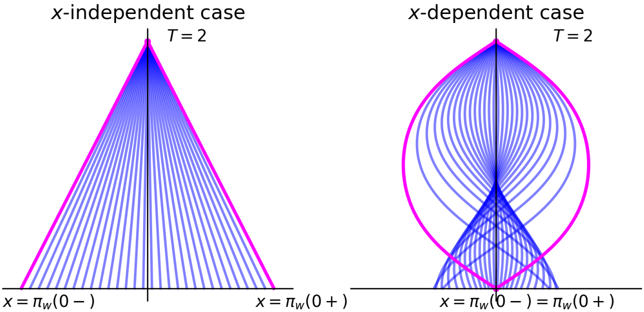
$$I_T(\omega) \neq \emptyset \iff \omega \in S_T(L^\infty(\mathbb{R})) \iff r_\omega^T \text{ nondecreasing a.e.} ⚠$$

And $I_T(\omega) \neq \emptyset \implies$

1. $I_T(\omega)$ *convex,*
2. $I_T(\omega)$ *is a cone,*
3. $I_T(\omega)$ *is a* F_σ *set for the* L^1_{loc} *topology.*

...

Oops



x-independent case
$T = 2$
$x = \pi_w(0-)$
$x = \pi_w(0+)$
x-dependent case
$T = 2$
$x = \pi_w(0-) = \pi_w(0+)$

Inverse design results: but not exactly

Theorem (Colombo-P.-Sylla)

There exists, $T > 0$, H *and* ω *such that*

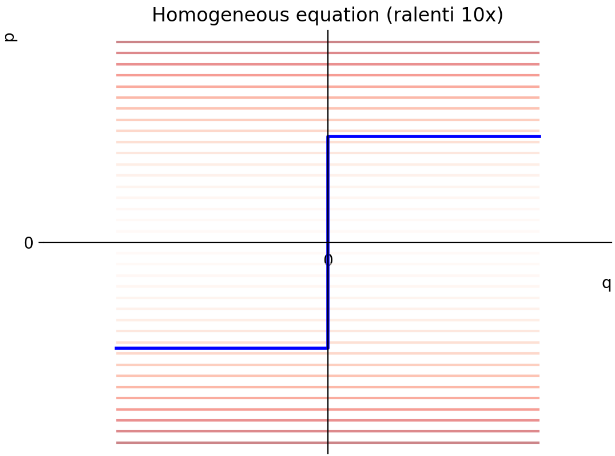
1. $I_T(\omega) \neq \emptyset$,
2. $\forall \rho_0 \in I_T(\omega), \exists (E, Q)$ *entropy-entropy flux pair, such that*

$$\partial_t E(\rho) + \partial_x \left(Q(x,\rho)\right) - E'(\rho)\partial_1 H(x,\rho) - \partial_1 Q(x,\rho) \neq 0$$

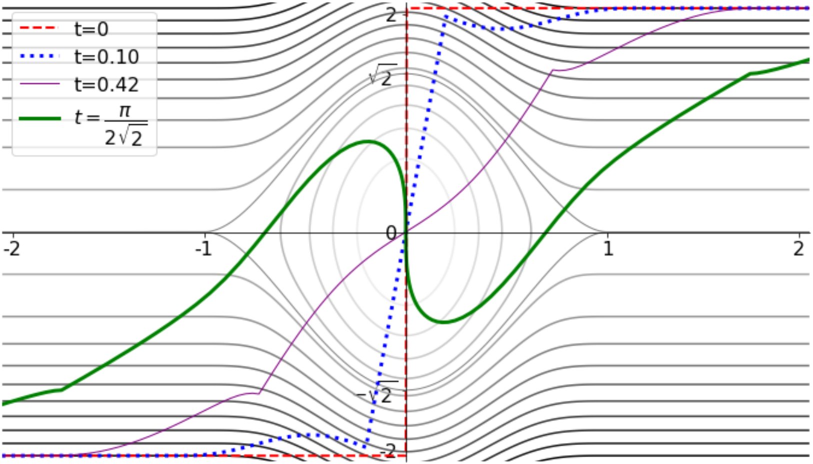
Remark

1. *Isentropic solutions* $``=''$ *closure of classical solutions.*
2. *Range of* S_t *homogeneous: just look at classical solutions.*
3. *Range of* S_t *nonhomogeneous: larger.*

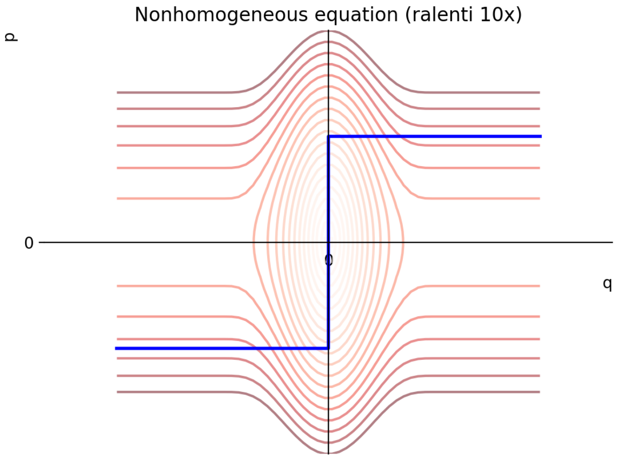
Rarefaction in the homogeneous case



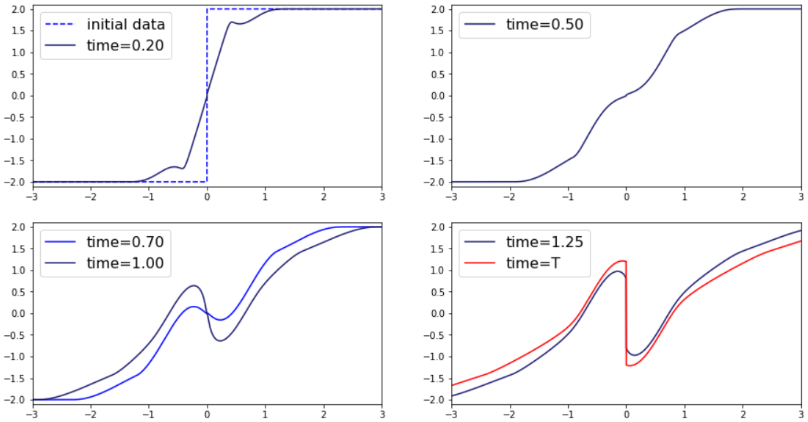
Hamiltonian Flow



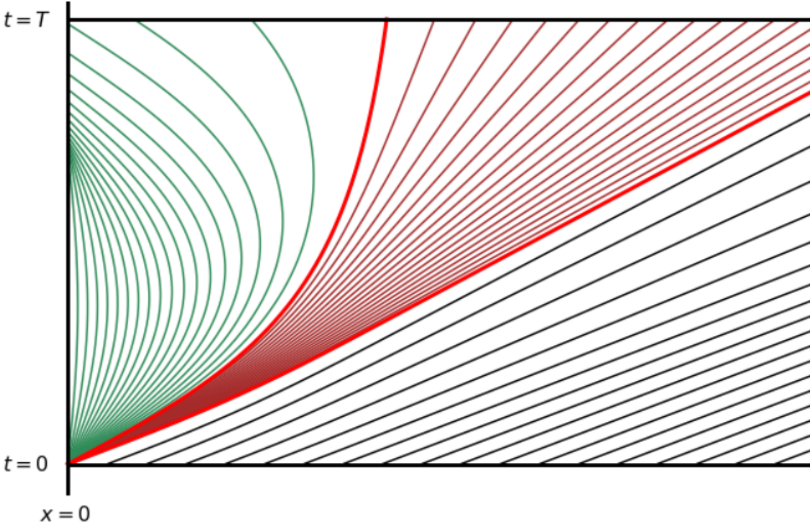
Rarefaction in the nonhomogeneous case



Solution of the PDE



The foward characteristics



Tool: Hamilton-Jacobi connection and optimal control

If $L(x,q) = \sup_{p \in R} (pq - H(x,p))$ (Legendre transform)

$$\begin{cases} V(t,x) := \inf_{c \in L^\infty(0,T)} \left(\int_0^t L(y(s), c(s)) ds + P(y(0)) \right) \\ \dot{y}(s) = c(s) \quad 0 < s < t \\ y(t) = x \end{cases}$$

$$\iff \begin{cases} \partial_t V + H(x, \partial_x V) = 0, \\ V(0) = P \\ V \text{ viscosity solution} \end{cases}$$

$$\overset{\rho = \partial_x V}{\iff} \begin{cases} \partial_t \rho + \partial_x (H(x, \rho)) = 0, \\ \rho(0) = P' \\ \rho \text{ entropy solution} \end{cases}$$

Open Problems

- Numerics.
- Nonconvex Flux.
  1. Generic flux: no more convexity of I_T!
  2. Still HJB but differential games, Isaacs condition?
- Multi dimensional conservation laws.
  1. No more HJB!
  2. Still entropy measure.
- Systems of conservation laws.

References

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3 C. Esteve et E. Zuazua, « The Inverse Problem for Hamilton-Jacobi Equations and Semiconcave Envelopes », SIAM J. Math. Anal., 2020.

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5 C. Esteve-Yagüe et E. Zuazua, « Differentiability With Respect to the Initial Condition for Hamilton-Jacobi Equations », SIAM J. Math. Anal, 2022.

6 R. M. Colombo, V. Perrollaz, A. Sylla « Initial data identification in space dependent conservation laws and Hamilton–Jacobi equations », Preprint, 2023.

THANK YOU FOR YOUR ATTENTION