Journée ANR
Paris 16–17 March 2023

Homogenization of Hamilton–Jacobi equations on networks.

A. Siconolfi, Università di Roma "La Sapienza".

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We use a variational method over suitable spaces of probability measures.

Literature

– Imbert, Monneau 2014

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Specified homogenization, periodic and stochastic with applications to traffic models.

- Galise, Imbert, Monneau 2015
- Forcadel and several coauthors

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The Hamiltonian is lifted to $\mathbb{R}^N \times \mathbb{R}^N$ by periodicity, and the following ϵ–problems are considered

$$\begin{cases} u^\epsilon_t + H(x/\epsilon, Du^\epsilon) = 0 & \text{in } \mathbb{R}^N \times (0,+\infty) \\ u^\epsilon = g & \text{in } \mathbb{R}^N \times \{0\}. \end{cases}$$

where g is a continuous, in general non periodic, initial datum.

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$$\begin{cases} u_t + \overline{H}(Du) = 0 & \text{in } \mathbb{R}^N \times (0, +\infty) \\ u = g & \text{in } \mathbb{R}^N \times \{0\}. \end{cases}$$

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- The effective Hamiltonian $\overline{H} : \mathbb{R}^N \to \mathbb{R}$ is the function which makes correspond to any p the (uniquely determined) value for which the cell problem

$$H(x, Dv + p) = \overline{H}(p)$$

admits a periodic solution.

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Namely

$$u(x,t) = \min \left\{ g(\xi(0)) + \int_0^t L(\xi, \dot{\xi})\, ds \right\}$$

where the minimum is over the curves with $\xi(t) = x$.

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The idea is to pass to the limit as $\epsilon \to 0$ in the formulas representing the solutions to the ϵ approximating problems.

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The most relevant output of our work is to show that this result can adapted to the framework of networks/graphs.

Mather's result

For periodic homogenization the ϵ–Hamiltonians are given by $H(x/\epsilon, p)$. However this formulation does not make sense on manifolds and the same on graphs/networks.

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The corresponding Lagrangians are $L(x, \epsilon q)$ and the action becomes

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The asymptotic problem related to the homogenization procedure is

$$\lim_{\epsilon \to 0} \left[\inf \epsilon \int_0^{\frac{T}{\epsilon}} L(\xi, \dot{\xi})\, dt \right]$$

Mather proved (1991) that the above limit does not exist, moreover the asymptotic behavior of the above value function does not depend on x and y, but instead, roughly speaking, on the rotations of the curves linking them.

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Equivalently: The variational problem must be therefore lifted to an appropriate space where such a construction is possible, in geometric jargon a covering manifold.

This is similar to what done in the periodic setting when the ϵ–oscillating problems are lifted from $\mathbb{T}^N$ to $\mathbb{R}^N$.

Networks

– A network $\mathcal{N} \subset \mathbb{R}^N$ can be understood as a piecewise regular 1– dimensional manifold. It has the form

$$\mathcal{N} = \bigcup_{\gamma \in \mathcal{E}} \gamma([0,1])$$

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– It is non oriented, namely for any arc γ, we also consider the inverse arc

$$\widetilde{\gamma}(t) = \gamma(1-t) \qquad \text{for } t \in [0,1].$$

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– Initial and final points of any arc γ, namely $\gamma(0)$ and $\gamma(1)$, have a special status, they are called vertices of the network. The set of vertices is denoted by $\mathbf{V}$. The key condition is that arcs with different support can intersect only at the vertices.

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- A curve on $\mathcal{N}$ is an absolutely function $\xi : [0, T] \to \mathcal{N}$. We will also consider special curves whose support is made by the union of the supports concatenated arcs.

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- The curve ξ is called a circuit if it is closed and injective in $(0, T)$.

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Hamiltonians and HJ equations on $\mathcal{N}$

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– We assume

- continuity in s, continuous differentiability in μ;
- convexity in μ;
- superlinearity in μ;
- the map $s \mapsto \min_\mu H_\gamma(s,p)$ is constant for any γ.

Note that the first three assumption are standard. The last one is necessary for the analysis of the corresponding stationary equation in Weak KAM setting. See

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A solution is a continuous function $v : \mathcal{N} \times (0, +\infty) \to \mathbb{R}$ such that

- $v(\gamma(s), t)$ is solution to (HJγ) for any γ;
- v satisfies suitable additional conditions on the discontinuity interfaces

$$\{(x,t) \mid x \in \mathbf{V},\ t \in (0, +\infty)\}$$

We have uniqueness of the solution once an initial continuous datum is prescribed at $t = 0$ and flux limiter c_x at any vertex x. The flux limiter plays an essential role in the conditions on the discontinuity interfaces

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From now on we take c_x as the minimal flux limiter.

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We obtain a lower semicontinuous Lagrangian $L(x, q)$ defined on the tangent bundle of $\mathcal{N}$ made up by elements of $(x, q) \in \mathcal{N} \times \mathbb{R}^N$ with q of the form

$$q = \rho \, \dot{\gamma}(s) \qquad \text{if } x = \gamma(s), \text{ with } \rho \in \mathbb{R}$$

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satisfying

$$L(x,0) = -c_x \qquad \text{at any vertex } x$$

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Covering networks

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– The vertices of the covering graph $\hat{\mathcal{N}}$ are $\mathbf{V} \times \mathbb{Z}^M$, where M is a dimension to be identified. $\hat{\mathcal{N}}$ is therefore embedded in $\mathbb{R}^N \times \mathbb{R}^M$. It is locally finite.

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- We will make precise later which vertices are linked by an arc. The arcs of $\hat{\mathcal{N}}$ are of the form (γ, η) where γ is an arc of $\mathcal{N}$ and η is a segment of $\mathbb{R}^M$ parametrized in $[0, 1]$.

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- We will make precise later which vertices are linked by an arc. The arcs of $\hat{\mathcal{N}}$ are of the form (γ, η) where γ is an arc of $\mathcal{N}$ and η is a segment of $\mathbb{R}^M$ parametrized in $[0,1]$.
- We lift the Hamiltonian by periodicity

$$H_{(\gamma,\eta)}(s,\mu) = H_\gamma(s,\mu) \qquad \text{for } (x,\mu) \in [0,1] \times \mathbb{R}.$$

Approximating and limit equations

The approximating equations are

$$u_t^\epsilon + H_{(\gamma,\eta)}(s, u'/\epsilon) = 0 \qquad (\mathrm{HJ}\epsilon)$$

with flux limiters

$$c_{(x,h)} = c_x \qquad \text{for any } (x,h) \in \mathbf{V} \times \mathbb{Z}^M$$

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The limit equation is posed in $\mathbb{R}^M$ and has the form

$$u_t + \bar{H}(Du) = 0 \qquad \text{(HJ)}$$

coupled with a continuous initial datum g. Here $\bar{H} : \mathbb{R}^M \to \mathbb{R}$ is an Hamiltonian to be identified.

Convergence of functions

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To help understanding this point, we remark that $\mathbb{R}^M$ and $\mathbf{V} \times \mathbb{Z}^M$ are isometric if we look at them at a large scale distance. Same as $\mathbb{Z}^M$ and $\mathbb{R}^M$.

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The map $F : \mathbf{V} \times \mathbb{Z}^M \to \mathbb{R}^M$ defined as

$$F(x, h) = h$$

is in fact a quasi–isometry, which is the equivalent of an isometry for large scale geometry.

Convergence of functions

We have to prove that the solutions of the approximating problems defined in $\hat{\mathcal{N}} \times [0, +\infty)$ converge in some sense to the solution of the limit equation defined in $\mathbb{R}^M \times (0, +\infty)$.

We have also to relate the initial data g_ϵ defined in $\mathcal{N}$ to g which is defined in $\mathbb{R}^M$.

To help understanding this point, we remark that $\mathbb{R}^M$ and $\mathbf{V} \times \mathbb{Z}^M$ are isometric if we look at them at a large scale distance. Same as $\mathbb{Z}^M$ and $\mathbb{R}^M$.

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Definition

A map F between two metric spaces (X, d_x), (Y, d_Y) is called a quasi–isometry if there exist $k \geq 1$ and $A \geq 0$ with

$$\frac{1}{k} d_X(x_1, x_2) - A \leq d_Y(F(x_1), F(x_2)) \leq k\, d_X(x_1, x_2) + A$$

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- A quasi–isometry is coarsely surjective, in the sense that for any $y \in Y$ there is an image $F(x)$ close to y
- and coarsely injective, in the sense that if $F(x_1) = F(x_2)$ then x_1 and x_2 are close.

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We say that a sequence $u_\epsilon : \mathcal{N} \to \mathbb{R}$ F_ϵ–locally uniformly converges to $u : \mathbb{R}^{b(\Gamma)} \to \mathbb{R}$ if for any subsequence $(x_{\epsilon_n}, h_{\epsilon_n})$ with

$$\epsilon_n \, h_{\epsilon_n} \to h$$

one has

$$u_{\epsilon_n}(x_{\epsilon_n}, h_{\epsilon_n}) \to u(h)$$

Identifying the dimension M

- We fix a maximal tree $\mathcal{T}$ in $\mathcal{N}$, namely a subnetwork of $\mathcal{N}$ without nontrivial cycles containing all the vertices of $\mathcal{N}$. Such an object does exists, even if it is not unique.

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M is equal to the number of elements of $\mathcal{E}^+ \setminus \mathcal{E}^+_{\mathcal{T}}$. It is called first Betti number of Γ and is an indicator of the topological complexity of the network.

The map θ

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There is an interpretation in terms of electricity flow. In the tree $\mathcal{T}$ there is no flow of electricity since there are no nontrivial circuits. However any arc γ outside $\mathcal{T}$, because of the maximality of $\mathcal{T}$, allows closing a circuit. The circuit is actually $\theta(\gamma)$.

Homology groups

Altogether we have defined a map

$$\theta : \mathcal{E}^+ \setminus \mathcal{E}^+_{\mathcal{T}} \rightarrow \text{ family of circuits of } \mathcal{T}$$

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Namely the group of formal sums of these circuits with coefficients in $\mathbb{Z}$, with the identification $\theta(\tilde{\gamma}) = -\theta(\gamma)$ so that the following cancellation law holds

$$\theta(\gamma) + \theta(\tilde{\gamma}) = 0$$

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Same construction can be performed with coefficients in $\mathbb{R}$ obtaining $H_1(\Gamma, \mathbb{R})$.

We complete the definition of θ through

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$H_1(\Gamma, \mathbb{Z}) \sim \mathbb{R}^M$ is the space where the limit equation of the homogenization procedure is posed

Rotation number of a curve

To any curve ξ with $\operatorname{spt}\xi = U_i \operatorname{spt}\gamma_i$ we associate the rotation number

$$\theta(\xi) = \sum_{i=1}^{M} \theta(\gamma_i) \in \mathbb{Z}^M.$$

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$$\inf \left\{ \int_0^T L(\xi, \dot{\xi})\, dt \mid \xi(0) = x,\, \xi(T) = y,\, \theta(\xi) = h \right\}$$

where $x, y \in \mathbf{V}$, $h \in \mathbb{Z}^M$.

We complete the definition of the covering network $\hat{\mathcal{N}}$ with vertices $\mathbf{V} \times \mathbb{Z}^M$ prescribing the two vertices (x_1, h_1), (x_2, h_2) are connected by an arc (γ, η) if

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- x_1 and x_2 are connected by γ in $\mathcal{N}$;
- $h_2 - h_1 = \theta(\gamma)$.

In this case

$$\eta(t) = (1 - t)\, h_1 + t\, h_2.$$

In this setting we have

Fact

The problem

$$\inf\left\{\int_0^T L(\xi,\dot\xi)\,dt \mid \xi(0)=x,\ \xi(T)=y,\ \theta(\xi)=h\right\}$$

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In this setting we have

Fact

The problem

$$\inf\left\{\int_0^T L(\xi,\dot{\xi})\,dt \mid \xi(0)=x,\ \xi(T)=y,\ \theta(\xi)=h\right\}$$

is equivalent to

$$\Phi((x_1,h_1),(x_2,h_2),T)=\inf\left\{\int_0^T L((xi,\eta),(\dot{\xi},\dot{\eta})\,dt\right\}$$

where the infimum is over the curves (ξ,η) *with* $(\xi(0),\eta(0))=(x,h_1)$, $(\xi(T),\eta(T))=(y,h_2)$, $h_2-h_1=h$.

Relaxed problems

We relax the above variational problem in a suitable space of measures.
To any curve ξ defined in $[0, T]$, we associate the occupation measure μ_ξ defined as

$$\mu_\xi(E) = \frac{1}{T} \int \chi_E(\xi, \dot{\xi})\, dt$$

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By relaxing the previous construction on curves, we can define a rotation vector $\rho(\mu) \in H_1(\mathcal{N}, \mathbb{R}) \sim \mathbb{R}^M$ for any closed measure.

We consider the problem

$$\inf \left\{ \int L(x, q)\, d\mu \right\}$$

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The above problem is well posed and there are minimizers The corresponding value function is denoted by $\beta : \mathbb{R}^M \to \mathbb{R}$ and is convex and superlinear.

Mather's result on networks

Mather's result on networks

Theorem

For any positive A, δ, there exists $T_0 = T_0(A, \delta)$ such that

$$\left| \frac{1}{T} \Phi((x_1, h_1)(x_2, h_2)T) - \beta\left(\frac{l-m}{T}\right) \right| < \delta$$

for $T > T_0$, and $\left|\frac{l-m}{T}\right| < A$.

The convex dual of the function β is the effective Hamiltonian $\bar{H}$ appearing in the limit problem of the homogenization procedure.

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For any $p \in \mathbb{R}^M$, $\bar{H}(p)$ is univocally defined as the value for which the stationary equation

$$H_\gamma(s, v' + p \cdot \theta(\gamma)) = \bar{H}(p)$$

has solution in $\mathcal{N}$.

Main result

Theorem

Assume that initial data g_ϵ F_ϵ locally converge to g, then the solutions u^ϵ of (HJϵ) *with initial datum g_ϵ F_ϵ locally converge to the solution u of* (HJ) *with initial datum g.*