Some optimal control problems on metric spaces: stratified systems & mean field control

Hasnaa Zidani
Joint work with **O. Jerhaoui**, F. Jean, A. Prost

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Outline

Hamilton-Jacobi-Bellman (HJB) approach

$$\vartheta(t,x) = \min\left\{\Phi(y(T)) \mid \dot{y}(s) = f(y(s),u(s)),\ \ y(t) = x, \quad u(s) \in U \text{ a.e.}\right\}$$

- ϑ satisfies the dynamic programming principle:

$$\begin{aligned}
\vartheta(t,x) &= \min_{u\in U} \vartheta(t+h, y^u_{t,x}(t+h)) \qquad h \in (0, T-t),\ x \in \mathbb{R}^d, \\
\vartheta(T,x) &= \Phi(x).
\end{aligned}$$

- ϑ is the unique bounded lsc (or continuous) *viscosity* solution of the HJB equation:

$$\begin{aligned}
&-\partial_t\vartheta(t,x) + \mathcal{H}(x, D_x\vartheta(t,x)) = 0 \qquad t \in [0,T[,\ \ x \in \mathbb{R}^d, \\
&\vartheta(T,x) = \Phi(x).
\end{aligned}$$

Here $\mathcal{H}(x,p) := \max_{u\in U}(-p \cdot f(x,u))$

Stratified systems

- Over the last decade, there has been an increasing interest in studying optimal control problems and HJ approach on networks or tree-like structures.
- These problems have a great impact in real-world applications:

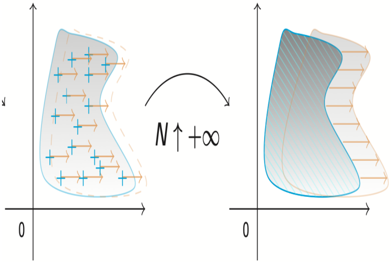
Figure: Vehicule traffic

Figure: Smart grids

- These structures can be modelled as a special class of geodesic spaces.

Multi-agent systems (or mean field control)

- An other problem that attracts many interest in optimal control is the multi-agent problem where the state lies in the space of probability measure $\mathcal{P}(M)$.



$$\begin{cases} \partial_t v(t,\mu) + H(\mu, D_\mu v) = 0, \ \mu \in \mathcal{P}(M) \\ v(T,\mu) = g(\mu). \end{cases}$$

Ref.: Bonnet, Rossi, Frankowska, Marigonda, Quincampoix, Cardaliaguet, Jimenez, Piccoli, ...

- The space of probability measures $\mathcal{P}(M)$ is also a geodesic space.

Hadamard spaces

- Let (X, d) be a complete metric space where:
  1. any two points x_0 and x_1 of X are connected by a constant speed speed geodesic, i.e. a map $\gamma : [0, 1] \to X$ such that
  $$\gamma_0 = x_0, \quad \gamma_1 = x_1, \quad d(\gamma_t, \gamma_s) = C|t - s|, \ \forall t, s \in [0, 1],$$
  2. and the following inequality is verified for any geodesic γ:
  $$d^2(y, \gamma_t) \leq (1 - t)d^2(y, \gamma_0) + td^2(y, \gamma_1) - t(1 - t)d^2(\gamma_0, \gamma_1),$$
  for every $y \in X$.
- Then (X, d) is called a Hadamard space[1].

[1] Hadamard spaces are also known as complete CAT(0) spaces

Examples of Hadamard spaces

- Euclidean spaces, Hilbert spaces.
- A *collection* of manifolds

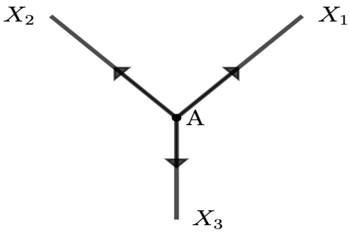


Figure: one-dimensional network

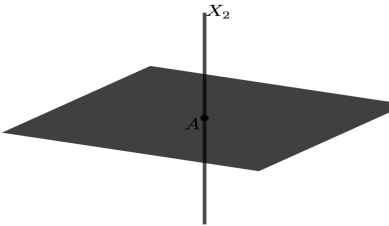
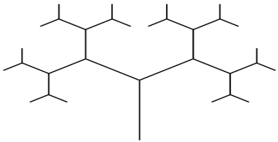


Figure: multi-dimensional network

- Metric $\mathbb{R}$-trees.



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- We consider the following stationary HJ equation

$$\begin{cases} H(u(x), x, D_x u) = 0, & \forall x \in \Omega, \\ u(x) = \ell(x), & \forall x \in \partial\Omega, \end{cases}$$

and the time dependent variant,

$$\begin{cases} \partial_t u + H(x, D_x u) = 0, \quad \forall\, (t, x) \in (0, +\infty) \times X, \\ u(0, x) = \ell(x), \quad x \in X, \end{cases}$$

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How to define the Hamiltonian? How to define viscosity notion?

How can we define the derivative $D_x v$?

Does a comparison principle hold in this setting?

Can we derive existence of the solution via Perron's method?

State of the art

HJ equations on networks ;

- Optimal control problems on 1d networks: Imbert-Monneau-HZ'13, Achdou-Tchou'15, Hermosilla-HZ'15, Lions-Souganidis'17, Morfe'18, Barles-Chasseigne'22
- Eikonal equation on Ramified spaces: Camilli, Marchi and Schieborn'15.

HJ equations on general metric spaces

- Eikonal type equations on a general metric space: Giga, Hamamuki and Nakayasu'22.
- A class of Hamilton Jacobi equations with Hamiltonians of the form $H(x, |D_x u|)$: Gangbo and ŚwiUech'21, Ambrosio and Feng'14.

Outline

Viscosity notion: DC functions

Definition (Semiconvex/semiconcave/DC functions)

Let $F : X \to \mathbb{R}$ be a function.

- We say that F is semiconvex if there exists $\lambda \in \mathbb{R}$ such that for every geodesic $\alpha : [0, 1] \to X$ the following inequality holds
$$F(\alpha_t) \le (1 - t)F(\alpha_0) + tF(\alpha_1) - \frac{\lambda}{2} t(1 - t)d^2(\alpha_0, \alpha_1).$$
- We say that F is semiconcave if $-F$ is semiconvex.
- We say that F is DC if it can be represented as a difference of semiconvex functions

Theorem

Let $F : X \to \mathbb{R}$ be a locally Lipschitz and DC function.

Then F admits directional derivatives everywhere. In the sequel, we will say that F is differentiable and we denote the differential by $D_x F$.

Examples of locally Lipschitz and DC functions

- The distance function to a fixed point $y \in X$

$$X \ni x \mapsto d(x, y)$$

is semiconvex.

- The squared distance function to a fixed point $y \in X$

$$X \ni x \mapsto d^2(x, y)$$

is semiconvex.

- The distance function to a closed convex subset of X, denoted by $C \subset X$

$$X \ni x \mapsto d(x, C),$$

is semiconvex.

- DC function are abundant in Hadamard spaces.

Viscosity notion

$$\begin{cases} H(u(x), x, D_x u) = 0, & \forall x \in \Omega, \\ u(x) = \ell(x), & \forall x \in \partial\Omega, \end{cases}$$ where $H : \mathbb{R} \times \mathrm{DC}(TX) \to \mathbb{R}$.

Definition (Viscosity solutions)

- An u.s.c function $u : \Omega \to \mathbb{R}$ is a viscosity **subsolution** if $\forall x \in \Omega$, and for all ϕ a loc. Lipschitz and **semiconvex** function s.t. $u - \phi$ attains a local **max** at x we have:

$$H(u(x), x, D_x\phi) \leq 0.$$

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- A continuous function $v : \overline{\Omega} \to \mathbb{R}$ is a viscosity solution, if it is both a supersolution and a subsolution and verifies the boundary condition

$$v(x) = \ell(x) \quad \forall x \in \partial\Omega.$$

Comparison principle

$$\begin{cases} H(u(x), x, D_x u) = 0, & \forall x \in \Omega, \\ u(x) = \ell(x), & \forall x \in \partial\Omega. \end{cases}$$

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where $H : \mathbb{R} \times \mathrm{DC}(TX) \to \mathbb{R}$ satisfies

- There exists $K > 0$ such that for all $\alpha > 0$, for all $r \in \mathbb{R}$ and for all $x, y \in \Omega$, we have

$$H(r, x, D_x(-\alpha d^2(., y))) - H(r, y, D_y(\alpha d^2(x, .))) \leq Kd(x, y)(1 + \alpha d(x, y)).$$

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- There exists $\gamma > 0$ such that

$$\gamma(r - s) \le H(r, x, p) - H(s, x, p) \quad \text{for all } r \ge s, \ x \in \Omega.$$

Comparison principle

Theorem (Comparison principle on bounded domains - Jerhaoui-HZ'22)

Let Ω be an open bounded set of X.
Consider $u : \overline{\Omega} \to \mathbb{R}$ is a bounded from above u.s.c subsolution, and $v : \overline{\Omega} \to \mathbb{R}$ is a bounded from below l.s.c supersolution.

If $u \leq v$ in $\partial\Omega$, then

$$u \leq v \quad \text{in } \overline{\Omega}.$$

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- It uses the variable doubling technique.
- A similar statement is true for the time dependent case.

Perron's method

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- In the classical case $X = \mathbb{R}^N$, Perron's method requires continuity of the Hamiltonian.
- In a general Hadamard space, we cannot assume such condition. Instead we assume the following:
  - For any $\phi : \Omega \to \mathbb{R}$ a semiconvex function, we have
$$(r, x) \mapsto H(r, x, D_x\phi)$$
is lower semicontinuous;
  - For any $\psi : \Omega \to \mathbb{R}$ a semiconcave function, we have
$$(r, x) \mapsto H(r, x, D_x\psi)$$
is upper semicontinuous;

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Theorem (Perron's method on a Hadamard space)

Let Ω be an open set of X.

– Assume the same assumptions as in the comparison theorem.

– Assume that the HJ equation admits a BC subsolution $\underline{u} : \overline{\Omega} \to \mathbb{R}$ and a BC supersolution $\overline{u} : \overline{\Omega} \to \mathbb{R}$

If

$$\underline{u}(x) \geq \ell(x) \geq \overline{u}(x), \quad \forall x \in \partial\Omega,$$

then there exists a unique BC viscosity solution of the HJB equation.

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- These results can be generalized to (locally) $\mathrm{CAT}(\kappa)$ spaces.

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- This unified framework is very convenient to study HJ equations on networks and stratified structures
- These results can be generalized to (locally) $\mathrm{CAT}(\kappa)$ spaces.
- Wasserstein spaces are almost **never** $\mathrm{CAT}(\kappa)$ spaces. However, these spaces have the flavor of spaces with curvature bounded from **below**. Jerhaoui-Jean-HZ'22, Jerhaoui-Prost-HZ (in preparation)