Microscopic derivation of a traffic flow model with a bifurcation

Pierre Cardaliaguet
Université Paris Dauphine

Based on joint works with N. Forcadel (INSA Rouen)
and on a work in progress with N. Forcadel, T. Girard and R. Monneau

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Two classical models of traffic flow

General goal: derive traffic flow models on junction from microscopic models.

Two kind of models for traffic flow on the line:

1) Microscopic models: e.g., the follow-the-leader model is a system of ODEs

$$\frac{d}{dt}U_i(t) = V(U_{i+1}(t) - U_i(t)),\ t \geqslant 0, \forall i \in \mathbb{Z}.$$

2) Macroscopic models: e.g., the Lighthill-Whitham-Richards (LWR) model is the scalar conservation law

$$\partial_t \rho + (\rho v(\rho))_x = 0 \qquad \text{in } \mathbb{R} \times (0, +\infty),$$

(M. J. Lighthill and G. B. Whitham (1955), P. I. Richards (1956))

Goal of the talk

- Discuss how to derive the LWR model

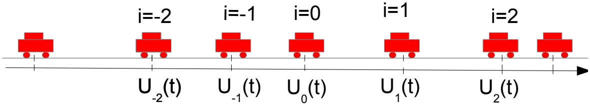
$$\partial_t \rho + (\rho v(\rho))_x = 0 \qquad \text{in } \mathbb{R} \times (0, +\infty),$$

  from the follow-the-leader model

$$\frac{d}{dt} U_i(t) = V(U_{i+1}(t) - U_i(t)), \; t \geqslant 0, \forall i \in \mathbb{Z}.$$

- Well-known when all the vehicles are identical and on a single road. Then $f(\rho) = \rho v(\rho) = \rho V(1/\rho)$ (Aw, Klar, Materne, and Rascle (2002))
- Main contributions: we address the case where
  - **the vehicles have a different behavior**
  - and **on a bifurcation**.

From an homogeneous traffic flow...



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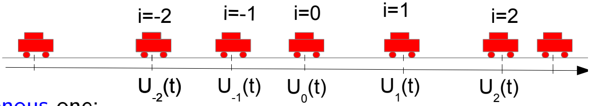
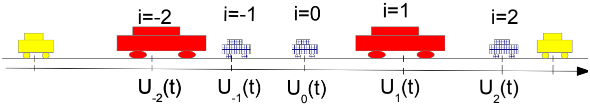


... to an heterogenous one:



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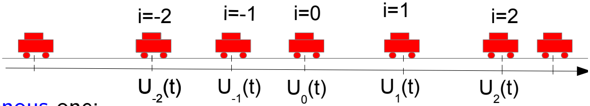

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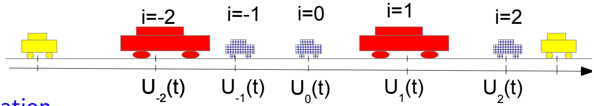

and with a bifurcation

Some references

Topic at the intersection of scalar conservation law, Hamilton-Jacobi and stochastic homogenization $\Longrightarrow$ Many many references!

Rigorous derivation of the macroscopic model from the microscopic one:

- For one type of vehicles: B. Argall, E. Cheleshkin, J. M. Greenberg, C. Hinde, and P.-J. Lin (2002), A. Aw, A. Klar, T. Materne, and M. Rascle (2002), M. Di Francesco and M. D. Rosini (2015), P. Goatin and F. Rossi (2017), H. Holden and N. H. Risebro (2018),..
- For several types of cars: N. Chiabaut, L. Leclercq, and C. Buisson (2010), N. Forcadel and W. Salazar (2015)

Analysis of microscopic models on a junction

- R.M. Colombo, H. Holden, and F. Marcellini (2020)

Analysis of macroscopic models on a junction

- Formulation in terms of conservation laws: G.D. Adimurthi, G. Veerappa (2003), E. Audusse, B. Perthame (2005), M. Garavello and B. Piccoli (2006), R. Burger, K.H. Karlsen, J. Towers (2009), M. Herty, J. P. Lebacque, and S. Moutari (2009), B. Andreianov, K.H. Karlsen, N.H. Risebro (2010), G. M. Coclite, M. Garavello, and B. Piccoli (2015), A. Bressan, and K.T. Nguyen (2015), B. Andreianov, and M. D. Rosini (2018), ..., M. Musch, U.S. Fjordholm and N.H. Risebro (2022)
- Formulation in terms of Hamilton-Jacobi equations: C. Imbert, R. Monneau, and H. Zidani (2013), G. Galise, C. Imbert, and R. Monneau, R. (2015), P.-L. Lions-P. Souganidis (2016, 2017), N. Forcadel, W. Salazar, and M. Zaydan (2018), N. Forcadel, and W. Salazar (2020),...

(Stochastic) homogenization of HJ equations, with works by Armstrong, Caffarelli, C., Ciomaga, Davini, Feldman, Kosygina, Lin, Lions, Nolen, Novikov, Papanicolau, Schwab, Seeger, Smart, Souganidis, Tran, Varadhan, Yilmaz, Zeitouni...
$\longrightarrow$ strongly inspired by the works of Kesten ('93) and Alexander ('93) in first passage percolation.

Outline

I) Traffic flow on the line

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II) Traffic flow on a bifurcation

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III) Link with conservation laws on junction

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A random microscopic model

We consider a random version of the follow-the-leader model:

$$\frac{d}{dt} U_i(t) = V_{Z_i}(U_{i+1}(t) - U_i(t)), \ t \geqslant 0, \forall i \in \mathbb{Z},$$

where

- Infinitely many cars, indexed by $i \in \mathbb{Z}$,
- U_i denotes the position of car i at time t,
- Cars are ordered: $U_i(t) \leqslant U_{i+1}(t)$ for all t, i,
- The velocity $V = V_{Z_i}(p)$ of car i depends on the distance p of car i to car $i+1$ and on the "type" Z_i of car i
- The types are (Z_i) are i.i.d. and take values in **a finite set** $\mathcal{Z}$.

(cf. N. Chiabaut, L. Leclercq, and C. Buisson ('10))

Assumptions

On the velocity map $V : \mathcal{Z} \times \mathbb{R}_+ \to \mathbb{R}$, we assume the following:

(H_1) For any $z \in \mathcal{Z}$, $p \to V_z(p)$ is Lipschitz continuous;

(H_2) For any $z \in \mathcal{Z}$, there exists $h_0^z > 0$ such that $V_z(p) = 0$ for all $p \in [0, h_0^z]$;

(H_3) For any $z \in \mathcal{Z}$, $p \to V_z(p)$ is increasing in $[h_0^z, +\infty)$;

(H_4) There exists $V_{\max} > 0$ and, for any $z \in \mathcal{Z}$, there exists $V_{\max}^z \leqslant V_{\max}$, such that $\lim_{p \to +\infty} V_z(p) = V_{\max}^z$.

The (integrated) distribution of cars

- For $\epsilon > 0$, we consider an initial condition (U_i^0) and let (U_i) be the solution of

$$\frac{d}{dt}U_i(t) = V_{Z_i}(U_{i+1}(t) - U_i(t)),\ t \geqslant 0, \forall i \in \mathbb{Z}.$$

  with initial condition (U_i^0).

- We are interested in the distribution of vehicles $\sum_{i\in\mathbb{Z}} \delta_{U^i(t)}$.

- (Integrated distribution) We set

$$N^\omega(x,t) = \sum_{i\in\mathbb{Z},\ i\leqslant 0} \delta_{U_i(t)}((x,+\infty)) - \sum_{i\in\mathbb{Z},\ i>0} \delta_{U_i(t)}((-\infty,x]).$$

  Remark: $\partial_x N^\omega(\cdot,t) = -\sum_{i\in\mathbb{Z}} \delta_{U_i(t)}$.

- (Scaled integrated distribution)

$$\nu^{\epsilon,\omega}(x,t) = \epsilon N^\omega(x/\epsilon, t/\epsilon) \qquad \forall (x,t) \in \mathbb{R}\times[0,+\infty).$$

Main result for the problem on the line

Theorem (C.-Forcadel, (SIMA '21))

- Assume that $\nu^{\epsilon,\omega}(\cdot, 0)$ converges locally uniformly and a.s. to the Lipschitz continuous map $\nu_0 : \mathbb{R} \to \mathbb{R}$.
- Then the $\nu^{\epsilon,\omega}$ converges a.s. and locally uniformly to the unique (Lipschitz) continuous viscosity solution ν to

$$\begin{cases} \partial_t \nu + \bar{H}(\partial_x \nu) = 0 & \text{in } \mathbb{R} \times]0, +\infty[\\ \nu(x, 0) = \nu_0(x) & \text{in } \mathbb{R} \end{cases}$$

- where the effective Hamiltonian $\bar{H}$ is given by $\bar{H}(p) = p\bar{V}(-1/p)$ with $\bar{V} : [0, +\infty) \to [0, \min_{z\in\mathcal{Z}} V^z_{\max}]$ defined by
  - $\bar{V}(p) = 0$ if $p \leqslant \bar{h}_0$ where $\bar{h}_0 := \mathbb{E}[h_0^{Z_0}]$,
  - and $\mathbb{E}[V_{Z_0}^{-1}(\bar{V}(p))] = p$ if $p > \bar{h}_0$.

Link with the Lighthill-Whitham-Richards (LWR) model

We come back to the (rescaled) empirical density of cars:

$$\rho^{\epsilon}(t) = \epsilon \sum_{i \in \mathbb{Z}} \delta_{\epsilon U_i^{\epsilon}(t/\epsilon)}, \qquad t \geqslant 0.$$

Corollary [Convergence to the LWR model]

As $\epsilon \to 0$, $\rho^{\epsilon}(t)$ converges, a.s., in distribution and locally uniformly in time, to

$$\rho(x,t) := -\partial_x \nu(x,t),$$

where ν is the solution of the limit HJ equation. Moreover ρ is the entropy solution of the LWR model

$$(LWR) \qquad \partial_t \rho + \partial_x(\rho \bar{v}(\rho)) = 0 \qquad \text{in } \mathbb{R} \times \mathbb{R}_+,$$

with initial condition $\partial_x \nu_0(\cdot)$ and where the fundamental diagram is given by $\bar{v}(\rho) = \bar{V}(1/\rho)$.

Ingredients of proof

- Localization argument: approximate finite speed of propagation,
- Existence of correctors,
- Standard techniques in homogenization of HJ equations

Approximate finite speed of propagation

Lemma

There exists $\beta > 0$ such that, if (U_i) and $(\tilde{U}_i)$ are two solutions of the equation with $U_i(0) \leqslant \tilde{U}_i(0)$ for $i \leqslant i_0$ (where $i_0 \in \mathbb{Z}$), then

$$U_i(t) \leqslant \tilde{U}_i(t) + 2^{(i-i_0)} e^{\beta t} \qquad \forall i \leqslant i_0,\ t \geqslant 0.$$

Consequence: comparison principle.

Existence of correctors

Given $\theta \in (0, \underline{V_{\max}})$, we consider the random sequence (c_i^θ) defined by

$$c_0^\theta = 0, \qquad c_{i+1}^\theta = c_i^\theta + V_{Z_i}^{-1}(\theta).$$

In other words,

$$V_{Z_i}(c_{i+1}^\theta - c_i^\theta) = \theta \qquad \forall i \in \mathbb{Z}.$$

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Central remark: The family $(\tilde{U}_i^\theta(t) := c_i^\theta + t\theta)_{i\in\mathbb{Z}}$ is a self-similar and "almost planar" solution to the system

$$\frac{d}{dt}U_i(t) = V_{Z_i}(U_{i+1}(t) - U_i(t)),\; t \geqslant 0, \forall i \in \mathbb{Z},$$

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Indeed

$$\frac{d}{dt}\tilde{U}_i^\theta(t) = \theta = V_{Z_i}(c_{i+1}^\theta - c_i^\theta) = V_{Z_i}(\tilde{U}_{i+1}^\theta(t) - \tilde{U}_i^\theta(t)).$$

while, by the law of large numbers, a.s.,

$$\frac{\tilde{U}_n^\theta(t)}{n} = \frac{c_n^\theta}{n} + \frac{t\theta}{n} = \frac{1}{n}\sum_{i=0}^{n-1} V_{Z_i}^{-1}(\theta) + \frac{t\theta}{n} \to \mathbb{E}\left[V_{Z_0}^{-1}(\theta)\right] \qquad \text{as } n \to \pm\infty.$$

Construction of the effective velocity

Recall that $\bar{h}_0 := \mathbb{E}[h_0^{Z_0}]$. Given $p > \bar{h}_0$, we consider the solution $\bar{U}^p$ to the problem with linear initial condition:

$$\frac{d}{dt}\bar{U}_i^p(t) = V_{Z_i}(\bar{U}_{i+1}^p(t) - \bar{U}_i^p(t)),\ t \geqslant 0, \qquad \bar{U}_i^p(0) = p\ i \qquad \forall i \geqslant 0.$$

Proposition [Convergence for linear initial conditions]

There exists $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that for every $p \geqslant 0$, $i \in \mathbb{N}$ and $\omega \in \Omega_0$

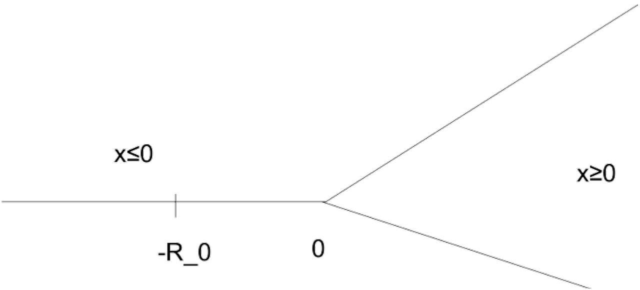
$$\lim_{t \to +\infty} \frac{\bar{U}_i^p(t)}{t} = \bar{V}(p) \qquad \forall i \geqslant 0,$$

where the continuous and non-decreasing map $\bar{V} : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

- $\bar{V}(p) = 0$ if $p \leqslant \bar{h}_0$ where $\bar{h}_0 := \mathbb{E}[h_0^{Z_0}]$,
- $\mathbb{E}[V_{Z_0}^{-1}(\bar{V}(p))] = p$ if $p > \bar{h}_0$.

Outline

The model on a bifurcation



We consider a traffic model with

- One incoming road,
- K outgoing roads,
- A bifurcation between $-R_0$ and 0,
- No overtaking (expect between $-R_0$ and 0),
- We set $\mathcal{R} = ((-\infty, 0] \times \{0\}) \cup \bigcup_{k=1}^{K} ([0, +\infty) \times \{k\})$.

Types and order

- Infinitely many cars, indexed by $i \in \mathbb{Z}$,
- The type of car i is denoted by Z_i,
  - The Z_i are i.i.d. with values in a finite set $\mathcal{Z}$,
- The outgoing road $T_i \in \{1, \dots, K\}$ chosen by car i is determined by Z_i: namely $T_i := T(Z_i)$ where $T : \mathcal{Z} \to \{1, \dots, K\}$,
- For $k \in \{1, \dots, K\}$, we set $\pi_k = \mathbb{P}[T_i = k]$: this is the proportion of cars taking road k.
- The cars are "ordered": the car in front of car i is
  - $i + 1$ if $U_i(t) < -R_0$,
  - ℓ_i if $U_i(t) > 0$, where $\ell_i = \inf\{j > i, \ T_j = T_i\}$,
  - $i + 1$ and ℓ_i if $U_i(t) \in [-R_0, 0]$.

Dynamics

The dynamics of the cars is given by

$$\frac{d}{dt} U_i(t) = V_{Z_i}\left(U_{i+1}(t) - U_i(t)\ ,\ U_{\ell_i}(t) - U_i(t)\ ,\ U_i(t)\right), \qquad t \geqslant 0,\ i \in \mathbb{Z}.$$

where

$$V_z(e_1, e_2, x) = \begin{cases} \tilde{V}_z^0(e_1) & \text{if } x \leqslant -R_0 \\ \tilde{V}_z^k(e_2) & \text{if } x \geqslant 0 \text{ and } k = T(z) \end{cases}$$

The (integrated) distribution of cars

- Fix an initial condition $(U_i^0)_{i\in\mathbb{Z}}$ and let U be the associate solution.
- (Integrated distribution) We set, for $k \in \{1,\dots,K\}$ and $(x,t) \in [0,+\infty) \times [0,+\infty)$,

$$N^\omega(x,k,t) = \sum_{i\in\mathbb{Z},\ i\leqslant 0,\ T_i=k} \delta_{U_i(t)}((x,+\infty)) - \sum_{i\in\mathbb{Z},\ i>0,\ T_i=k} \delta_{U_i(t)}((-\infty,x]).$$

  and for $x \leqslant 0$

$$N^\omega(x,0,t) = \sum_{i\in\mathbb{Z}} \delta_{U_i(t)}((x,+\infty)) - \sum_{i\in\mathbb{Z},\ i>0,\ T_i=k} \delta_{U_i(t)}((-\infty,x]).$$

- (Scaled integrated distribution)

$$\nu^{\epsilon,\omega}(x,k,t) = \begin{cases} \epsilon(\pi^k)^{-1} N^\omega(x/\epsilon,k,t/\epsilon) & \forall (x,k,t) \in [0,+\infty) \times \{1,\dots,K\} \times [0,+\infty) \\ \epsilon N^\omega(x/\epsilon,0,t/\epsilon) & \forall (x,t) \in (-\infty,0] \times (-\infty,0] \end{cases}$$

Assumptions

(H_1) For any $z \in \mathcal{Z}$, the map $(e_1, e_2, x) \to V_z(e_1, e_2, x)$ is Lipschitz continuous from $\mathbb{R}_+^2 \times \mathbb{R}$ to $\mathbb{R}_+$ and nondecreasing with respect to the first two variables;

(H_2) There exists $e_{\max} > \Delta_{\min} > 0$ and $0 < R_2 < R_1 < R_0$, with $R_0 > e_{\max}$, such that for any $z \in \mathcal{Z}$, for any $(e_1, e_2, x) \in \mathbb{R}_+^2 \times \mathbb{R}$,

- (i) $V_z(e_1, e_2, x) = 0$ if $(e_1 \leqslant \Delta_{\min}$ and $x \leqslant -R_2)$ or if $(e_2 \leqslant \Delta_{\min}$ and $x \geqslant -R_1)$,
- (ii) $V_z(e, e_2, x) = V_z(e_{\max}, e_2, x)$ and $V_z(e_1, e, x) = V_z(e_1, e_{\max}, x)$ if $e \geqslant e_{\max}$;

(H_3) There exists $\tilde{V}^0, \dots, \tilde{V}^K : [0, +\infty) \to [0, +\infty)$ such that

$$V_z(e_1, e_2, x) = \begin{cases} \tilde{V}_z^0(e_1) & \text{if } x \leqslant -R_0 \\ \tilde{V}_z^k(e_2) & \text{if } x \geqslant 0 \text{ and } T(z) = k. \end{cases}$$

(H_4) For any $z \in \mathcal{Z}$ and any $k \in \{0, \dots, K\}$, there exists $h^k_{\max,z} \in (\Delta_{\min}, e_{\max}]$ such that $p \to \tilde{V}_z^k(p)$ is increasing and concave in $[\Delta_{\min}, h^k_{\max,z}]$ and constant on $[h^k_{\max,z}, +\infty)$;

(H_5) There exists $\kappa > 0$ such that, for any $z \in \mathcal{Z}$,

- (i) $V_z(e_1, e_2, x) = \tilde{V}_z^0(e_1)$ if $e_1 \leqslant e_2$, $x \leqslant -R_2$ and $V_z(e_1, e_2, x) \leqslant \kappa$,
- (ii) $\partial_x V_z(e_1, e_2, x) \geqslant 0$ if $x \in [-R_1, 0]$ and $V_z(e_1, e_2, x) \leqslant \kappa$,
- (iii) $V_z(e_1, e_2, x) > 0$ if $e_1 \wedge e_2 > \Delta_{\min}$.

Main convergence result

Theorem (C.-Forcadel (To appear in ARMA))

Under the previous assumptions on V, there exists a constant $\bar{A} < 0$ (the flux limiter) such that, if $\nu^\epsilon(\cdot, \cdot, 0)$ converges locally uniformly in $\mathcal{R}$ and a.s. to a Lipschitz continuous map $\nu_0 : \mathcal{R} \to \mathbb{R}$, then ν^ϵ converges locally uniformly and a.s. in $\mathcal{R} \times [0, +\infty)$ to the unique continuous viscosity solution of the Hamilton-Jacobi equation with flux limiter $\bar{A}$:

$$\begin{cases} \partial_t \nu(x,k,t) + \bar{H}^k(\partial_x \nu(x,k,t)) = 0 \qquad \text{in } (\mathcal{R}\backslash\{0\}) \times (0,+\infty) \\ \partial_t \nu + \max\{\bar{A}, \bar{H}^{0,+}(\partial_0 \nu), \bar{H}^{1,-}(\partial_1 \nu), \dots, \bar{H}^{K,-}(\partial_K \nu)\} = 0 \text{ at } x = 0 \\ \nu(x,k,0) = \nu_0(x,k) \qquad \text{in } \mathcal{R}. \end{cases}$$

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The homogenized Hamiltonian $\bar{H}^k$: Let $\bar{V}^k$ be the homogenized velocities on the single road k. We have set

$$\bar{H}^0(p) = p\bar{V}^0(-1/p), \ \bar{H}^k(p) = p\bar{V}^k(-1/(\pi^k p)) \qquad p \in (-\infty, 0)$$

Viscosity solutions: test functions are continuous on $\mathcal{R}$ and C^1 on each branch.

Notation: $\bar{H}^{0,+}$ (resp. $\bar{H}^{k,-}$ is the largest nondecreasing map below $\bar{H}^0$ (resp. the largest nonincreasing map below $\bar{H}^k$).

Ingredients of proof

- Existence and uniqueness (by comparison) of the viscosity solution on a junction are due to Imbert-Monneau ('13).
  (see also Lions-Souganidis ('16) and the monograph by Barles-Chasseigne ('18))

- Localization argument: extension of the "almost" finite speed of propagation,
  $\Longrightarrow$ any limit (up to a subsequence) of ν^ϵ solves the HJ outside the junction.

- Main difficulty: construction of the flux limiter $\bar{A}$

Construction of $\bar{A}$

- We assume for simplicity that $\bar{H}^0 = \bar{H}^1 = \cdots = \bar{H}^K =: \bar{H}$.

- Let $e > 0$ be such that $\bar{H}(-1/e) = \min_p \bar{H}(p)$ and let $(U_{e,i})$ be the solution with initial condition $U_{e,i}(0) = ei$ for $i \in \mathbb{Z}$.

- The time function: For $t \geqslant 0$, let $\theta_e(t)$ be the number of vehicle having gone through 0 between time 0 and time t:

$$\theta_e(t) = \sharp \left\{ i \in \mathbb{Z},\ \exists s \in [0,t] \text{ with } \ U_{e,i}(s) = 0 \right\}.$$

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Theorem (Limit of the time function)

The limit $\bar{\vartheta}_e$ of $\theta_e(t)/t$ exists a.s. as $t \to +\infty$, with $\bar{\vartheta}_e \leqslant -\min H^0$. The flux limiter is then $\bar{A} := -\bar{\vartheta}_e$.

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Main steps of proof:

- A concentration inequality: There exists $C > 0$ such that

$$\mathbb{P}\left[\left|\theta_e(t) - \mathbb{E}\left[\theta_e(t)\right]\right| \geqslant \epsilon t\right] \leqslant C \exp\{-\epsilon^2 t/C\}.$$

- Superadditivity property: for $\tilde{h} < -\min_p \bar{H}(p)$, set $\bar{M}_{e,\tilde{h}}(t) = \inf_{s\in[0,t]} \mathbb{E}\left[\theta_e(s)\right] - \tilde{h}s$. Then

$$\bar{M}_{e,\tilde{h}}(t_1 + t_2) \geqslant \bar{M}_{e,\tilde{h}}(t_1) + \bar{M}_{e,\tilde{h}}(t_2) - C(1 + (\ln(t_1 + t_2))^{1/8}(t_1 + t_2)^{7/8}).$$

Outline

I) Traffic flow on the line

II) Traffic flow on a bifurcation

III) Link with conservation laws on junction

The conservation law on the network

Recall that the limit flow ν solves the Hamilton-Jacobi equation

$$\begin{cases} \partial_t \nu(x,k,t) + \bar{H}^k(\partial_x \nu(x,k,t)) = 0 & \text{in } (\mathcal{R}\backslash\{0\}) \times (0,+\infty) \\ \partial_t \nu + \max\{\bar{A}, \bar{H}^{0,+}(\partial_0 \nu), \bar{H}^{1,-}(\partial_1 \nu), \dots, \bar{H}^{K,-}(\partial_K \nu))\} = 0 \text{ at } x = 0 \\ \nu(x,k,0) = \nu_0(x,k) & \text{in } \mathcal{R}. \end{cases}$$

Let us set

$$\rho(t,x,k) := -\pi^k \partial_x \nu(t,x,k).$$

Then (outside of the junction) ρ is an entropy solution of the scalar conservation law

$$\partial_t \rho(t,x,k) + (f^k(\rho(t,x,k))_x = 0 \qquad \text{in } (0,+\infty) \times \overset{o}{\mathcal{R}},$$

where $f^k(v) = -\pi^k \bar{H}^k(-v/\pi^k)$.

Lemma [Detailed Rankine-Hugoniot condition]

We have, for any $k \in \{1,\dots,K\}$,

$$f^k(\rho(t,0,k)) = \pi^k f^0(\rho(t,0,0)) \qquad \text{a.e. } t > 0.$$

$\longrightarrow$ One needs however stronger conditions to select the solution (Adimurthi, Mishra and Veerappa Gowda ('03))

Stationary solutions

For the Hamilton-Jacobi, stationary solutions are of the form

$$u(t,x,k) = -(\pi^k)^{-1} e_k x - tH^k(-e_k/\pi^k)$$

where $e = (e_0, \ldots, e_K) \in \mathbb{R}^{K+1}$ and

1. **(continuity at $x = 0$)** $H^k(-e_k/\pi^k) = H^0(-e_0)$, i.e.,

$$f^k(e_k) = \pi^k f^0(e_0) \qquad \forall k \in \{1, \ldots, K\}.$$

2. **(condition on the junction)** we have $H^0(-e_0) \geqslant \bar{A}$ and

$$\text{either } H^0(-e_0) = \bar{A}, \text{ or } H^{0,+}(-e_0) = H^0(-e_0),$$
$$\text{or } H^{k,-}(-(\pi^k)^{-1}e_k) = H^k(-(\pi^k)^{-1}e_k) \text{ for some } k \in \{1, \ldots, K\}.$$

   which can be written as:

$$f^0(e_0) \leqslant -\bar{A} \text{ and either } f^0(e_0) = -\bar{A}, \text{ or } f^{0,+}(e_0) = f^0(e_0),$$
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- Such $e \in \mathbb{R}^{K+1}$ should correspond to the stationary solutions of the conservation law on the junction.
- Following [Andreianov, Karlsen, and Risebro, '11] and [Musch, Fjordholm and Risebro, '22] we define the germ $\mathcal{G}$ as the set $e \in \mathbb{R}^{K+1}$ satisfying condition (1) and (2).

The notion of germs

Following [Andreianov, Karlsen, and Risebro, '11] and [Musch, Fjordholm and Risebro, '22],

- a solution ρ to the conservation law on the junction associated with the germ $\mathcal{G}$ is an entropy solution on $(0,+\infty)\times \overset{o}{\mathcal{R}}$ satisfying

$$(*) \qquad \rho(t,0)\in\mathcal{G} \qquad \text{a.e. } t>0.$$

  ($\rho(t,0)$ is the trace of ρ in the sense of Panov ('07))

- Existence, uniqueness and stability (L^1 contraction) of solutions are proved when $\mathcal{G}$ satisfies the Rankine-Hugoniot condition and is "mutually consistent" and "maximal":
  - The germ $\mathcal{G}$ is mutually consistent if for any $U=(u^j), \bar{U}=(\bar{u}^j)\in G$,

$$q^0(u^0,\bar{u}^0)\geqslant\sum_{j=1}^{K}q^j(u^j,\bar{u}^j)$$

    where $q^j(c',c):=(f^j(c')-f^j(c))\mathrm{sign}(c'-c)$ $(j=0,\dots K)$.
  - The germ $\mathcal{G}$ is maximal if, for any $U=(u^j)$ satisfying the Rankine-Hugoniot condition,

$$\left[\; q^0(u^0,\bar{u}^0)\geqslant\sum_{j=1}^{K}q^j(u^j,\bar{u}^j) \qquad \forall\bar{U}=(\bar{u}^j)\in\mathcal{G}\;\right] \qquad\Longrightarrow\qquad U\in\mathcal{G}.$$

Analysis of our germs

Recall that the germs arising in our analysis if given by

$$\begin{aligned}\mathcal{G} = &\{U = (u^j), \text{ such that } u^j = \pi^j u^0 \qquad \forall j = 1, \dots K,\\ &\qquad f^0(u^0) \leqslant -\bar{A} \text{ and } \Big[\text{ either } f^0(u^0) = -\bar{A}, \text{ or } f^{0,+}(u^0) = f^0(u^0),\\ &\qquad \text{or } f^{k,-}(u^k) = f^k(u^k) \text{ for some } k \in \{1, \dots, K\} \Big].\end{aligned}$$

Lemma

- If $K = 1$, the germ $\mathcal{G}$ is mutually consistent and maximal.
- If $K \geqslant 2$, the germ $\mathcal{G}$ is **not** mutually consistent in general.

Theorem [C.-Forcadel-Girard-Monneau]

If $K = 1$ and u solves HJ, then $\rho := -\partial_x u$ satisfies (*).

Idea of proof:

- "Standard" outside the junction (Caselles ('92), Colombo-Perrollaz-Sylla ('22))
- By discretization (numerical schemes) on the junction
- ... or by approximation by very smooth data.

Summary and open problems

In this talk we have

- derived (in terms of HJ eq) the macro behavior of cars on a bifurcation from its micro behavior
- made the link with conservation laws with discontinuous flux when $K = 1$

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- Obtain a complete relationship between formulations in terms of conservation law and Hamilton-Jacobi equations when $K \geqslant 2$
- Models with several lines, incoming and outgoing roads, overtake...

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Thank you!