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Mathématiques & Physique Théorique

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Some ideas to solve perilous problems for viscosity solutions : equations with discontinuities

(or how to advertise a forthcoming book...)

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(Book in collaboration with E. Chasseigne,
inspired by a lot of very interesting works
written by many very clever people)

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What is our aim ?

In standard control problems, we are given, for every $x \in \mathbb{R}^N$ and $t \in (0, T)$, a set $BL(x, t)$ of couples (b, l) of admissible velocities and the associated costs. The state of the system and the running cost are then described by the solution of the differential inclusion

$$(\dot{X}(s), \dot{L}(s)) = (b(s), l(s)) \in BL(X(s), t - s) \ ,$$
$$(X(0), L(0)) = (x, 0) \ ,$$

Under general assumptions, one can solve this ode and the value function is defined by

$$\mathbb{U}(x, t) = \inf_{(X,L)} \{L(t) + u_0(X(t))\} \ .$$

where $u_0 \in C(\mathbb{R}^N)$ is the final cost.

What is our aim ?

The Hamiltonian is given by

$$H(x,t,p_x) = \sup_{(b,l)\in BL(x,t)} \left(-b\cdot p_x - l\right) \ ,$$

and we hope that the following result is true

Theorem : The value function $\mathbb{U}$ is continuous and the unique solution of the Hamilton-Jacobi-Bellman Equation

$$u_t + H(x,t,D_x u) = 0 \quad \text{in } \mathbb{R}^N \times (0,T) \ ,$$

with the initial data

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N \ .$$

Moreover we have comparison and stability results for the HJB-Equation.

What is our aim ?

This result is valid if H is continuous and if there exists a constant C and a modulus of continuity m such that, for all, $x, y \in \mathbb{R}^N$, $t, s \in [0, T]$, $p_x, q_x \in \mathbb{R}^N$:

$$|H(x,t,p_x) - H(y,s,p_x)| \leq C(|x-y| + m(|t-s|))(1+|p_x|) \, ,$$

$$|H(x,t,p_x) - H(x,t,q_x)| \leq C|p_x - q_x| \, .$$

For $BL(x, t)$, this means that the b, l are bounded and that (in a certain sense) b is Lipschitz in x, continuous in t (cf. Cauchy-Lipschitz) and l is continuous in x, t.

Using the notion of viscosity solution, one can prove that the above result is valid.

What is our aim ?

AIM : to treat cases where H and BL have discontinuities.

PROBLEM : All the theory strongly relies on these assumptions on H which clearly excludes the possibility of having discontinuities.

ALREADY KNOWN IF H and/or u ARE DISCONTINUOUS :

(i) The definition of subsolution

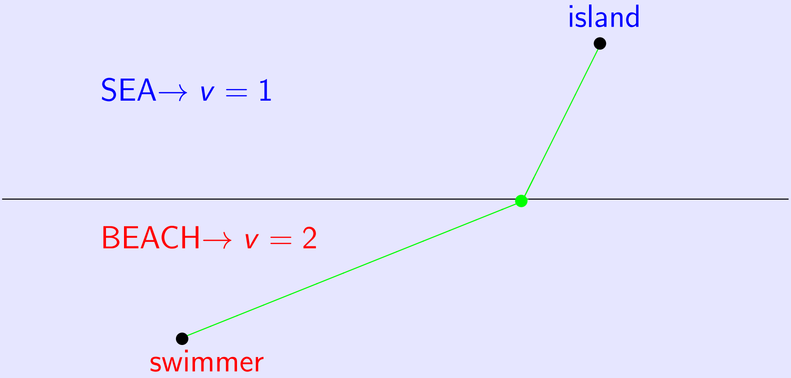
$$u_t^* + H_*(x, t, D_x u^*) \leq 0 \ ,$$

(ii) The definition of supersolution

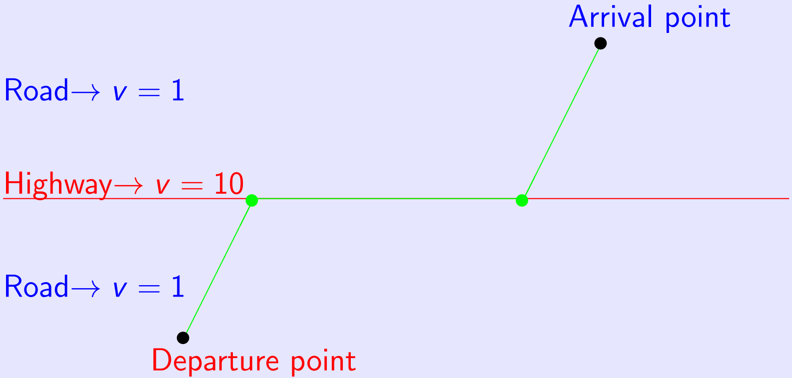
$$(u_*)_t + H^*(x, t, D_x u_*) \geq 0 \ ,$$

(iii) The half-relaxed limit method.

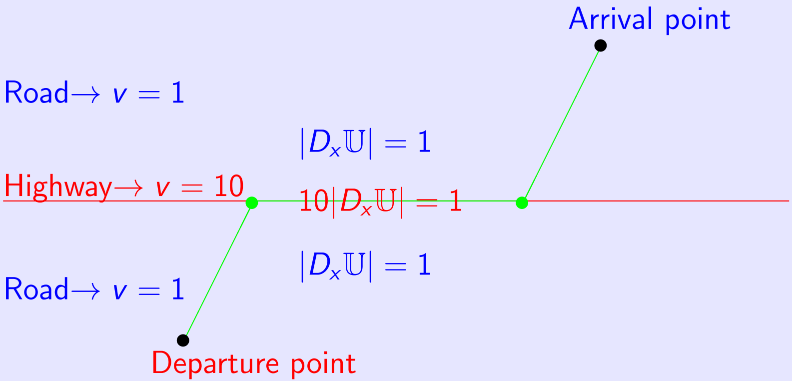
Some examples : the swimmer problem



Some examples : the highway problem (I)

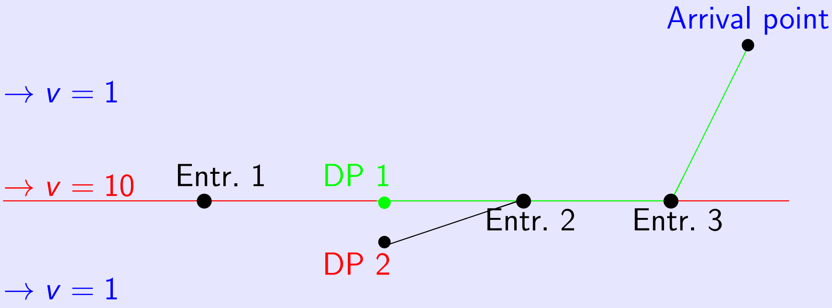


Some examples : the highway problem (I)



The classical subsolution condition does not see the highway !

Some examples : the highway problem (II)



Clearly $\mathbb{U}(DP2) \gg \mathbb{U}(DP\ 1)$: we have a discontinuity for $\mathbb{U}$ since we cannot immediately enter the highway at any point.

Conclusion

These examples show us that

(i) The subsolution condition causes a problem on each discontinuity : we will have to super-impose a suitable subsolution inequality on each discontinuity.

(ii) Normal controllability, i.e. to have dynamics which allows to reach quickly a discontinuity (or to leave it), seems necessary if the aim is to have a continuous value function.

And we add a “reasonnable” condition

(iii) Tangential continuity : outside the discontinuities, and at least in the directions which are parallel to the discontinuity, the Hamiltonien should satisfy the “good usual assumptions”.

Which kind of discontinuities ?

Whitney Stratification (Bressan-Hong revisited)

The discontinuities of H and BL – in x – are described by a partition of $\mathbb{R}^N$

$$\mathbb{R}^N = \tilde{\mathbf{M}}^0 \cup \tilde{\mathbf{M}}^1 \cup \cdots \cup \tilde{\mathbf{M}}^N ,$$

where

(i) for all k, $\tilde{\mathbf{M}}^k$ is a k-dimensional submanifold of $\mathbb{R}^N$,

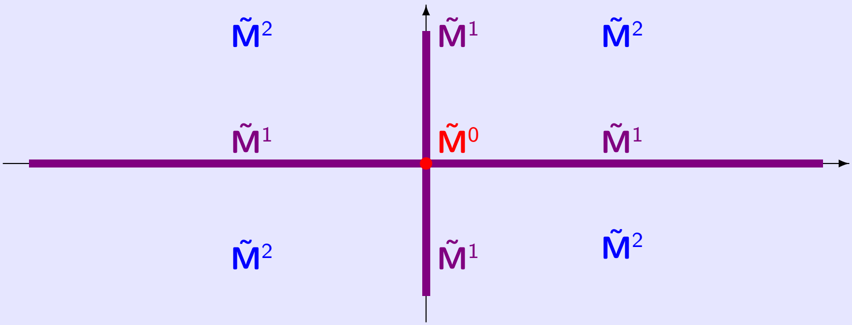
(ii) if $x \in \tilde{\mathbf{M}}^k$, there exists a neighborhood $\mathcal{V}_x$ of x which does not contain any point of $\tilde{\mathbf{M}}^0, \cdots, \tilde{\mathbf{M}}^{k-1}$,

(iii) If there exists two connected components $\tilde{\mathbf{M}}^k_i \cap \overline{\tilde{\mathbf{M}}^l_j} \neq \emptyset$ for two indices such that $l > k$ then $\tilde{\mathbf{M}}^k_i \subset \overline{\tilde{\mathbf{M}}^l_j}$.

$+$ specific assumptions (flat or “well-adapted” stratifications)

Which kind of discontinuities ?

Typical example in $\mathbb{R}^2$:



Stratified Solutions

We set $\mathbf{M}^k = \tilde{\mathbf{M}}^{k-1} \times (0, T)$ for $1 \leq k \leq N+1$ and

$$H^k(x,t,p_x) := \sup_{\substack{(b,l)\in \mathbf{BL}(x,t) \\ b \in T_x \tilde{\mathbf{M}}^{k-1}}} \Big\{ -b \cdot p - l \Big\} \quad \text{if } (x,t) \in \mathbf{M}^k.$$

Definition :

1. A stratified supersolution is a l.s.c function $v : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ such that
$$v_t + H(x,t,D_x v) \geq 0 \quad \text{in } \mathbb{R}^N \times (0,T).$$
2. A weak stratified subsolution is an u.s.c function $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ such that, for all k :
$$u_t + H^k(x,t,D_x u) \leq 0 \quad \text{on } \mathbf{M}^k.$$
3. A strong stratified subsolution is an u.s.c function $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ which satisfies, in addition
$$u_t + H_*(x,t,D_x u) \geq 0 \quad \text{in } \mathbb{R}^N \times (0,T).$$

Few words about control

Theorem : Under the "good assumptions of the stratified framework", the value function $\mathbb{U}$ is a strong stratified solution (= supersolution + strong stratified subsolution) of the HJB-Equation.

Proof :

(i) The supersolution inequalities are very classical.

(ii) On the contrary, the subsolution ones are far from being obvious on the $\mathbf{M}^k$ (since they are wrong in general...) : they are a consequence of the normal controllability.

Comparaison Result

Theorem : Under the "good assumptions of the stratified framework", if u is a weak, regular stratified subsolution and if v is a stratified supersolution such that $u(x,0) \leq v(x,0)$ in $\mathbb{R}^N$, we have

$$u(x,t) \leq v(x,t) \quad \text{in } \mathbb{R}^N \times [0,T).$$

Moreover, the result remains valid if u is a strong stratified subsolution.

NB : A subsolution is regular if, for all $k \leq N$, for all $(x,t) \in \mathbf{M}^k$

$$u(x,t) = \limsup\{u(y,s),\ (y,s) \to (x,t),\ (y,s) \in \mathbf{M}^{k+1} \cup \cdots \cup \mathbf{M}^{N+1}\}.$$

And

Strong subsolution $\Rightarrow$ Regular subsolution

Proof

(i) Ingredient 1 : Reduction to the proof of a "very local" comparaison result in $B(x,r) \times (t, t+h)$ where $r, h > 0$ are as small as we wish : this allows to reduce to the case when the discontinuities are affine subspaces.

(ii) Ingredient 2 :The proof is done by backward induction (!) on the co-dimension the discontinuity to which (x,t) belongs : the result is true if $(x,t) \in \mathbf{M}^{N+1}$, then we prove it for $(x,t) \in \mathbf{M}^{N}$, then for $(x,t) \in \mathbf{M}^{N-1}$...etc.

Proof

(iii) Ingredient 3 : The proof of the comparaison in $B(x,r) \times (t, t+h)$ if $(x,t) \in \mathbf{M}^k$ starts by the tangential regularisation of the subsolution : we obtain a sequence $(u_\varepsilon)_\varepsilon$ of Lipschitz continuous subsolutions which are C^1 on $\mathbf{M}^k$ –hence u_ε is a test-function on $\mathbf{M}^k$–.

This step uses, in a crucial way, the regularity of the subsolution, the normal controllability and the tangential continuity.

Proof

(iii) Ingredient 4 : the "Magical Lemma". If $(\bar{x}, \bar{t})$ is a maximum point of $u_\varepsilon - v$ then

– either the optimal trajectory for v in $(\bar{x}, \bar{t})$ goes away to the discontinuity $\mathbf{M}^k$ and we conclude using the sub and super-dynamic programming principle for u_ε and v.

= Control type argument

– or it goes to $\mathbf{M}^k$ then we have $v_t + H^k(\bar{x}, \bar{t}, D_x v) \geq 0$ holds and we conclude since u_ε is a test-function on $\mathbf{M}^k$.

= PDE type argument

Few words on the stability

The normal controllability plays an essential role in the stability properties. To pass to the limit in the inequalities $u_t + H^k(x, t, D_x u) \leq 0$ on $\mathbf{M}^k$ is rather delicate if one wants to take into account perturbations on the $\mathbf{M}^k$.

We can handle the three main cases

(i) The case when the perturbed stratification $(\mathbf{M}^k_\varepsilon)_k$ is just a smooth perturbation of $(\mathbf{M}^k)_k$.

(ii) The case when new discontinuities appears (hence we have new parts of $\mathbf{M}^k$ compared to $\mathbf{M}^k_\varepsilon$).

(iii) The case when some discontinuities disappears (hence $\mathbf{M}^k$ is a smaller set compared to $\mathbf{M}^k_\varepsilon$).

Some generalizations

(i) We can take into account time-dependent stratifications.

(ii) We can take into account more general equations (stationary or time-dependent) (for example with some unbounded control features).

(iii) We can take into account certain cases when the value function,is only l.s.c, "à la Barron-Jensen".

(iv) The state constraint case does not cause too many difficulties, even with "stratified boundary" (except for the regularity of subsolutions on the boundary of the domain which is an issue...).

Other results/applications

(v) Study of the equivalence between classical (sub)solutions and stratified (sub)solutions, with the double aim of generalizing the already known comparison results to less regular cases but also to be able to use standard stability results (half-relaxed limit method) : problems with boundary conditions (Dirichlet, Neumann, mixed...), application to KPP,...etc.

(vi) Reformulation in the "stratified" setting of non standard problem : the tanker problem.

(vii) Applications to various problems : large time behavior, homogeneization, fronts propagation or to concrete problems.

Other questions/Open problems

(i) Problems with jumps where the non local feature clearly induces some difficulty...

(ii) The case of "stratified network" : partially treated but more is needed !

(iii) Main open problem : a complete PDE proof of the differents results with a dream of the extension to non-convex Hamiltoniens...

Thank you for you attention

and

Do not hesitate to ask a lot of questions !