Comp. & Maths. with Appls., Vol. 5, pp. 349–357
Pergamon Press Ltd., 1979. Printed in Great Britain

INTERPOLATION ERROR ESTIMATES ON HERMITE RATIONAL "WACHSPRESS TYPE" THIRD DEGREE FINITE ELEMENT

J. L. Gout

Département de Mathématiques, Université de Pau, France

Communicated by E. L. Wachspress

(*Received April* 1979)

Abstract—This paper is about the study of interpolation error for the Hermite rational "Wachspress type" third degree finite element that is constructed in[1]. We obtain results analogous with those of the "corresponding" ADINI (*polynomial*) finite element.

1. INTRODUCTION

The notation used here is analogous to that in the papers of Apprato–Arcangeli–Gout [2], Gout[3], Wachspress[3]. More generally, for finite element notation see Ciarlet[4], Ciarlet-Raviart [5].

1.1. *Notation*

Consider $n \in \mathbf{N}^*$, Ω a bounded open non empty subset of $\mathbf{R}^n$ (with euclidean norm $\|\cdot\|$) and p a number such that $1 \leqslant p \leqslant +\infty$. For each $m \in \mathbf{N}$, the Sobolev space $W^{m,p}(\Omega)$ consists of those (class of) functions $v \in L^p(\Omega)$ for which all partial derivatives $\partial^\alpha v = (\partial^{|\alpha|}v)/(\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n})$ with $|\alpha| \leqslant m$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$, belong to the space $L^p(\Omega) \cdot W^{m,p}(\Omega)$ is equipped with the norm

$$\|v\|_{m,p,\Omega} = \left(\sum_{0 \leqslant |\alpha| \leqslant M} \int_\Omega |\partial^\alpha v(x)|^p \, \mathrm{d}x \right)^{1/p}.$$

We shall also use the semi-norms

$$|v|_{l,p,\Omega} = \left(\sum_{|\alpha| = m} \int_\Omega |\partial^\alpha v(x)|^p \, \mathrm{d}x \right)^{1/p}, \qquad l = 0, 1, \ldots, m,$$

and

$$[v]_{l,p,\Omega} = \left(\int_\Omega \|D^l v(x)\|^p \, \mathrm{d}x \right)^{1/p}, \qquad l = 0, 1, \ldots, m,$$

when $p < +\infty$, with the usual change when $p = +\infty$. For each $v \in W^{m,p}(\Omega)$, for each $l = 0, 1, \ldots, m$ and almost everywhere in Ω, $D^l v(x)$ denotes the lth derivative at a point $x \in \Omega$ with $D^l u(x) = u(x)$ for $l = 0$. Moreover, we replace the conventional $\mathring{K}$ by K in the compact spaces $W^{m,p}(\mathring{K})$ and also write $\|v\|_{m,p,k}$, $|v|_{l,p,K}$ and $[v]_{l,p,K}$ instead of $\|v\|_{m,p,\mathring{K}}$, $|v|_{l,p,\mathring{K}}$ and $[v]_{l,p,\mathring{K}}$.

Also, for each integer $k \geqslant 0$ and for each non empty subset A of $\mathbf{R}^n$, $P_k(A)$ is the vectorial space formed by the restriction to the set A of the polynomial functions with n variables of kth degree with respect to these n variables. For other notation and definitions relating to finite elements, see [2, 4, 6, 7].

1.2. *Review of Hermite rational third degree finite elements*

In this paragraph we recall the principal characteristics of the Hermite rational "Wachspress type" third degree finite element which is constructed in [1].

Let K be a closed convex quadrilateral in $\mathbf{R}^2$ which is neither a trapezium nor a parallelogram. The vertices a_i, $i \in I = \mathbf{Z}/_{4\mathbf{Z}}$, of K are labelled so that a_i and a_{i+1} are consecutive and a_4 is the most distant vertex from the exterior diagonal, d, of the quadrilateral.

Also, h_K (respectively ρ_K) is the diameter of K (respectively the supremum of the diameters of the spheres inscribed in K) and $\mathbf{R}^2$ is equipped with the euclidean distance δ.

For each $i \in I$, l_i is an element of $P_1(\mathbf{R}^2)$ so that $l_i(x) = 0$ is an equation of the straight line d_i through points a_{i-1} and a_i; α_1 denotes the intersection of the interior diagonals of K, α_2 (respectively α_3) the intersection of the straight lines d_1 and d_3 (respectively d_2 and d_4).

Moreover l is an element of $P_1(\mathbf{R}^2)$ such that $l(x) = 0$ is the equation of the exterior diagonal d, and $\delta(x, d)$ denotes the euclidean distance from x to the straight line d.

Let P_K be the vectorial space generated by the functions w_i^0, $w_{i,i-1}^1$ and $w_{i,i+1}^1$, for each $x \in K$ and for each $i \in I$ by

$$
\begin{aligned}
w_i^0(x) &= \frac{l(a_i)}{l_{i+2}(a_i)l_{i+3}(a_i)\gamma_i(a_i)} \quad \frac{l_{1+2}(x)l_{i+3}(x)\gamma_i(x)}{l(x)} \\
w_{i,i-1}^1(x) &= \frac{l(a_i)}{l_{i+1}(a_{i-1})l_{i+2}(a_i)l_{i+3}^2(a_i)} \quad \frac{l_{i+1}(x)l_{i+2}(x)l_{i+3}^2(x)}{l(x)} \\
w_{i,i-1}^1(x) &= \frac{l(a_i)}{l_{i+2}^2(a_i)l_{i+3}(a_i)l_i(a_{i+1})} \quad \frac{l_{i+2}^2(x)l_{i+3}(x)l_i(x)}{l(x)}.
\end{aligned}
\tag{1.1}
$$

Let Σ_K be the set, defined by

$$
\Sigma_K = \{v \mapsto v(a_i),\ i \in I;\ v \mapsto Dv(a_i)\cdot(a_j - a_i),\ i \in I,\ j = i-1, i+1\}. \tag{1.2}
$$

Then (K, P_K, Σ_K) is a third degree ($P_K \supset P_3(K)$ and $P_K \not\supset P_4(K)$) finite element which also is of class C^0 [1].

2. A GENERAL INTERPOLATION RESULT

In this paragraph we obtain a result which is an extension of the result on Hermite interpolation error proved as Theorem 2-1 in [7]. Here, the proof is analogous to the one used for Theorem 5-1 in [2].

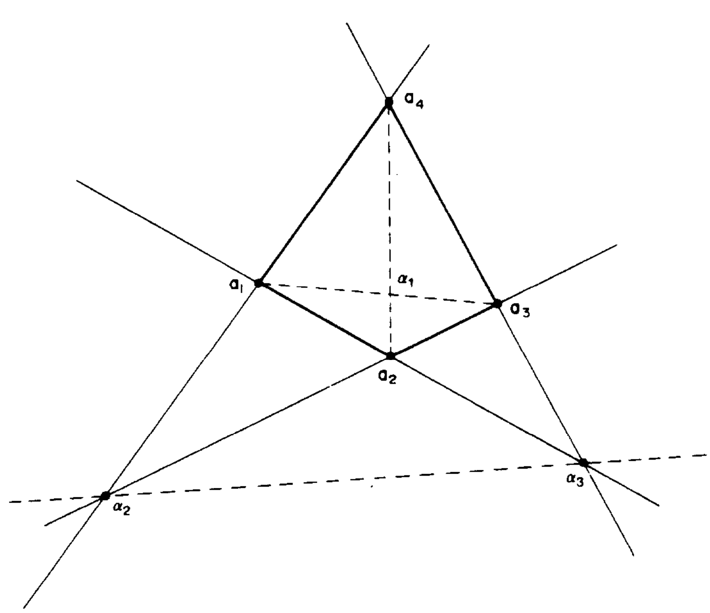

Fig. 1.

2.1. *Notation and assumptions*

First, we recall some specific notation for Hermite interpolation. Given a bounded nonempty subset Ω in $\mathbf{R}^n$, let s be an integer such that $1 \leq s \leq k$, P a finite-dimensional space ($\dim P = N$) of functions defined over the set $\bar{\Omega}$ and Σ a finite set of linear forms over $C^{k+1}(\bar{\Omega})$, linearly independent, defined from a set $\{a_i^r\}_{(0 \leq r \leq s)/(1 \leq i \leq N_r)}$ of points of $\bar{\Omega}$ (the nodes of the finite element), where N_r is the number of nodes when r is fixed,

$$\Sigma = \{\varphi_i^0,\ 1 \leq i \leq N_0;\ \varphi_{ij}^1,\ 1 \leq r \leq s,\ 1 \leq i \leq N_r,\ 1 \leq j \leq d_{ir}\}$$

with

$$\varphi_i^0;\ v \mapsto \varphi_i^0(v) = v(a_i) \qquad 1 \leq i \leq N_0$$

$$\varphi_{ij}^r;\ v \mapsto \varphi_{ij}^r(v) = D^r v(a_i^r) \cdot \xi_{ij}^r,\ 1 \leq r \leq s,\ 1 \leq i \leq N_r,\ 1 \leq j \leq d_{ir}$$

where d_{ir} denotes the number of the linear forms, corresponding to the nodes a_i^r and where the $\xi_{ij}^r(\xi_{ij}^r \in (\mathbf{R}^n)^r)$ are constructed from the geometry of the set of nodes. Clearly it follows

$$N = N_0 + \sum_{r=1}^{S} \left[\sum_{i=1}^{N_r} d_{ir}\right].$$

When Σ is P-unisolvent in the sense of Ciarlet [4], we recall that (K, P, Σ) denotes the generic finite element.

If v is a sufficiently smooth function, defined over $\bar{\Omega}$ and Πv the P-interpolant of v with respect to Σ, then

$$\Pi v = \sum_{i=1}^{N_0} \varphi_i^0(v) p_i^0 + \sum_{r=1}^{s} \sum_{i=1}^{N_r} \sum_{j=1}^{d_{ir}} \varphi_{ij}^r(v) p_{ij}^r,$$

where $\beta = \{p_i^0,\ 1 \leq i \leq N_0;\ p_{ij}^r,\ 1 \leq r \leq s,\ 1 \leq i \leq N_r,\ 1 \leq j \leq d_{ir}\}$ defines the set of basis functions of P with respect to Σ, and Π is the Hermite P-interpolation operator relative to Σ. In the notation of paragraph 1.1., we make the following assumptions

$$k + 1 > s + \frac{n}{P}. \tag{2.1}$$

Ω is a bounded nonempty open subset of $\mathbf{R}^n$, with a Lipschitz-continuous boundary, such that $\bar{\Omega}$ is stellated with regard to any node a_i^r of the finite element. (2.2)

Σ is a P-unisolvent set and $P_k(\bar{\Omega}) \subset P \subset C^{k+1}(\bar{\Omega})$. (2.3)

2.2. *Expression of the theorem*

THEOREM 2.1

In addition to the assumptions (2.1), (2.2) and (2.3), let Π be the Hermite P-interpolation operator relative to Σ, and let β be the set of basis functions of P with respect to Σ.

Then, for each $v \in W^{k+1,p}(\Omega)$, we obtain for each $m = 0, 1, \ldots, k$,

$$[v - \Pi v]_{m,p\Omega} \leq \frac{1}{k!} \frac{1}{k + 1 - \frac{n}{p}} \left\{\sum_{i=1}^{N_0} [p_i^0]_{m,\infty,\Omega}\right) [v]_{k+1,p,\Omega} h^{k+1}$$

$$+ \sum_{r=1}^{s} \frac{1}{(k-r)!} \frac{1}{k + 1 - r - \frac{n}{p}} \left(\sum_{i=1}^{N_r} \sum_{j=1}^{d_{ir}} [p_{ij}^r]_{m,\infty,\Omega}((\xi_{ij}^r))\right) [v]_{k+1,p,\Omega} h^{k+1-r}$$

and

$$[v-\Pi v]_{k+1,p,\Omega} \leqslant \frac{1}{k!}\ \frac{1}{k+1-\frac{n}{p}}\Big(\sum_{i=1}^{N_0}[p_i^0]_{k+1,\infty,\Omega}\Big)h^{k+1} + \Big(\sum_{i=1}^{N_0}[p_i^0]_{k,\infty,\Omega}\Big)h^k\quad [v]_{k+1,p,\Omega} \tag{2.4}$$

$$+\sum_{r=1}^{s}\frac{1}{(k-r)!}\left\{\frac{1}{k+1-r-\frac{n}{p}}\Big(\sum_{i=1}^{N_r}\sum_{j=1}^{d_{ir}}[p_{ij}^r]_{k+1,\infty,\Omega}\Big)((\xi_{ij}^r))h^{k+1-r} + \sum_{i=1}^{N_r}\sum_{j=1}^{d_{ir}}[p_{ij}^r]_{k,\infty,\Omega}\Big)((\xi_{ij}^r))h^{k-r}\right\}[v]_{k+1,p,\Omega}$$

where $((\xi_{ij}^r)) = \|\xi_{ij,1}^r\| \ldots \|\xi_{ij,r}^r\|$ with $\xi_{ij}^r = (\xi_{ij,1}^r, \ldots, \xi_{ij,r}^r) \in (\mathbf{R}^n)^r$

Proof. For $m = 0, 1, \ldots, k$ see Gout [7]. Let $m = k+1$. It suffices to prove the result for $v \in C^{k+1}(\bar{\Omega})$ and $1 < p < \infty$ (The proof simplifies for $p = 1$ and we may obtain the proof for $p = +\infty$ by a limiting procedure.)

From the identity (*cf.* p. 414 [7]):

$$\forall x \in \Omega \qquad D^k(\Pi v)(x) - D^k v(x) = \frac{1}{k!}\sum_{i=1}^{N_0} J_0(v, a_i^0)(x) D^k p_i^0(x)$$
$$+\sum_{r=1}^{s}\frac{1}{(k-r)!}\Big\{\sum_{i=1}^{N_r}\sum_{j=1}^{d_{ir}}(J_r(v, a_i^r)(x)\cdot\xi_{ij}^r)D^k p_{ij}^r(x)\Big\},$$

we obtain, by derivation, for each $\xi \in \mathbf{R}^n$,

$$D^{k+1}(\Pi v - v)(x)\cdot\xi = \frac{1}{k!}\sum_{i=1}^{N_0}[(DJ_0(v, a_i^0)(x)\cdot\xi)D^k p_i^0(x) + J_0(v, a_i^0)(x)D^{k+1}p_i^0(x)\cdot\xi]$$

$$+\sum_{r=1}^{s}\frac{1}{(k-r)!}\sum_{i=1}^{N_r}\sum_{j=1}^{d_{ir}}[(DJ_r(v, a_i^r)(x)\cdot(\xi_{ij}^r, \xi)D^k p_{ij}^r(x) + J_r(v, a_i^r)(x)\xi_{ij}^r D^{k+1}p_{ij}^r(x)\cdot\xi].$$

If we take the derivative of relation (2.5) of [7], we get

$$DJ_0(v, a_i^0)(x)\cdot\xi = -D^{k+1}v(x)\cdot((a_i^0 - x)^k, \xi)$$
$$DJ_r(v, a_i^r)(x)\cdot(\xi_{ij}^r, \xi) = -D^{k+1}v(x)\cdot((a_i^r - x)^{k-r}, \xi),$$

so that

$$\|D^{k+1}(\Pi v - v)(x)\| \leqslant \frac{1}{k!}\sum_{i=1}^{N_0}[\|D^{k+1}v(x)\|h^k\|D^k p_i^0(x)\| + \|J_0(v, a_i^0)(x)\|\quad\|D^{k+1}p_i^0(x)\|]$$

$$+\sum_{r=1}^{s}\frac{1}{(k-r)!}\sum_{i=1}^{N_r}\sum_{j=1}^{d_{ir}}[\|D^{k+1}v(x)\|h^{k-1}((\xi_{ij}^r))\|D^k p_{ij}^r(x)\|] + \|J_r(v, a_i^r)(x)\|((\xi_{ij}^r))\|D^{k+1}p_{ij}^r(x)\|,$$

and, with Proposition 2.1 of [7], we obtain the result.

REMARK 2.1.

Generally, the vectors ξ_{ij}^r (relations (2.12) of [7] are such that

$$\forall i,\ 1 \leqslant i \leqslant N_r, \qquad \forall j, \qquad 1 \leqslant j \leqslant d_{ir} \qquad ((\xi_{ij}^r)) \leqslant h^r. \tag{2.5}$$

Thus we can also show, using an analogous proof to the preceding one, the following estimates for each $m = 0, 1, \ldots, k$:

$$|v - \Pi v|_{m,p,\Omega} \leqslant \binom{n+m-1}{m}^{1/p}\left\{\frac{1}{k!}\frac{1}{k+1-\frac{n}{p}}\Big(\sum_{i=1}^{N_0}|p_i^0|_{m,\infty,\Omega}\Big)[v]_{k+1,p,\Omega}h^{k+1}\right.$$

$$+\sum_{r=1}^{s}\frac{1}{(k-r)!}\frac{1}{k+1-r-\frac{n}{p}}\left(\sum_{i=1}^{N_r}\sum_{j=1}^{d_{ir}}|p_{ij}^r|_{m,\infty,\Omega}\right)[v]_{k+1,p,\Omega}h^{k+1}\Bigg\}\ ,$$

and

$$|v-\pi v|_{k+1,p,\Omega}\leqslant\binom{n+k}{k+1}^{1/p}\left\{\frac{1}{k!}\left[\frac{1}{k+1-\frac{n}{p}}\left(\sum_{i=1}^{N_0}|p_i^0|_{k+1,\infty,\Omega}\right)h^{k+1}+\sum_{i=1}^{N_0}|p_i^0|_{k,\infty,\Omega}h^k\right][u]_{k+1,\infty,\Omega}\right.$$

$$\left.+\sum_{r=1}^{s}\frac{1}{(k-r)!}\left[\frac{1}{k+1-r-\frac{n}{p}}\left(\sum_{i=1}^{N_r}\sum_{j=1}^{d_{ir}}|p_{ij}^r|_{k+1,\infty,\Omega}\right)h^{k+1}+\left(\sum_{i=1}^{N_r}\sum_{j=1}^{d_{ir}}|p_{ij}^r|_{k,\infty,\Omega}\right)h^k\right][u]_{k+1,\infty,\Omega}\right\}. \qquad (2.6)$$

3. INTERPOLATION ERROR ESTIMATES

3.1. *The object of the analysis*

Let Ω be a bounded polygonal open subset in $\mathbf{R}^n$, let $\mathscr{H}$ be a set of positive numbers (redundant) and let $(\tau_h)_{h\in\mathscr{H}}$ be a regular family of $\bar{\Omega}$-triangulations by quadrilaterals K, which are non-degenerate into trapezia or parallelograms, whose diameters h_K are such that $h_K\leqslant h$.

We make the following assumptions [2]:

$$\exists\nu>0,\quad\forall h\in\mathscr{H},\quad\forall K\in\tau_h,\quad\inf_{x\in K}\delta(x,d)\geqslant\nu h_K \qquad (3.1)$$

(no degeneracy into a triangle).

$$\exists\sigma>o,\quad\forall h\in\mathscr{H},\quad\forall K\in\tau_h,\quad\frac{h_K}{\rho_K}\leqslant\sigma. \qquad (3.2)$$

(regular family of triangulations in the sense of Ciarlet–Raviart [5]).

Then the problem is to obtain interpolation error estimates over Ω with Hermite rational third degree finite elements (K, P_K, Σ_K).

First, we recall a preliminary result which is given in [2, 8]:

LEMMA 3.1.

Assume that the family of $\bar{\Omega}$ – triangulations $(\tau_h)_{h\in\mathscr{H}}$ satisfies assumption (3.1). Given $r\in\mathbf{N}$ and w a function such that for each $x\in K$

$$w(x)=\frac{\psi(x)}{l(x)}, \qquad (3.3)$$

where $\psi\in P_r(\mathbf{R}^2)$ and l is an element of $P_1(\mathbf{R}^2)$ as defined in Paragraph 1.2, then, for each $n\in\mathbf{N}^*$, there exists a constant $\mathscr{C}(m,r,\nu)>0$ such that

$$\forall h\in\mathscr{H},\quad\forall K\in\tau_h,\quad|w|_{m,\infty,K}\leqslant\frac{\mathscr{C}(m,r,\nu)}{\rho_K{}^m}\|w\|_{0,\infty,K}. \qquad (3.4)$$

3.2. *Interpolation error estimates*

At once we get:

PROPOSITION 3.1.

It is assumed that the family of triangulations $(\tau_h)_{h\in\mathscr{H}}$ satisfies assumptions (3.1) and (3.2). For $h\in\mathscr{H}$ and $K\in\tau_h$, let (K, P, Σ) be a Hermite rational "Wachspress type" third degree finite element. Then for each basis function w of P with respect to Σ, there exists a constant $\mathscr{C}_w(\nu)>0$, independent of h and K, such that

$$\|w\|_{0,\infty,K}\leqslant\mathscr{C}_w(\nu). \qquad (3.5)$$

Proof.

(1) *Basis functions*, w_i^0. Consider one of the basis functions, w_i^0, defined in (1.1). We have

$$\|w_i\|_{0,\infty,K} = \operatorname*{Sup}_{x\in K}\left|\frac{l(a_i)}{l_{i+2}(a_i)l_{i+3}(a_i)\gamma_i(a_i)}\,\frac{l_{i+2}(x)l_{i+3}(x)\gamma_i(x)}{l(x)}\right|$$

From results obtained for basis functions of the first degree Serendip finite element [2, 9] with

$$\operatorname*{Sup}_{x\in K}\left|\frac{l(a_i)}{l_{i+2}(a_i)l_{i+3}(a_i)}\,\frac{l_{i+2}(x)l_{i+3}(x)}{l(x)}\right| = 1,$$

we have

$$\|w_i\|_{0,\infty,K} \leqslant \operatorname*{Sup}_{x\in K}\left|\frac{\gamma_i(x)}{\gamma_i(a_i)}\right|, \tag{3.6}$$

and with propositions (3.2) and (3.4) of [2] we get

$$\operatorname*{Sup}_{x\in K}\left|\frac{\gamma_i(x)}{\gamma_i(a_i)}\right| \leqslant \left(1+\frac{1}{\nu}\right)^2 \operatorname*{Sup}_{\hat{x}\in\hat{K}}\left|\frac{\hat{\gamma}_i(\hat{x})}{\hat{\gamma}_i(\hat{a}_i)}\right|. \tag{3.7}$$

Now consider, for example, the basis function corresponding to point a_2 (the results would be analogous for the other vertices). If τ denotes the translation so that

$$\hat{K} = \tau(\tilde{K})$$

where $\tilde{K}$ is the square $[0,2]\times[0,2]$ and if $\hat{\gamma}_2$ is defined by relations (2.3) of [1], $\hat{a}_2 = \tau(\tilde{0},\tilde{0})$, we get

$$\frac{(\hat{\gamma}_2{}^0\tau)(\tilde{x})}{(\gamma_2{}^0\tau)(\hat{a}_2)} = \tilde{\alpha}\tilde{x}_1{}^2 + \tilde{\beta}\tilde{x}_2{}^2 + \tilde{\delta}\tilde{x}_1 + \tilde{\epsilon}\tilde{x}_2 + 1 \tag{3.8}$$

where $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\delta}$ and $\tilde{\epsilon}$ are dependent on α_2, β_2, δ_2 and ϵ_2 introduced in the formulas (2.3) of [1].

From (3.6) and (3.7) we obtain

$$\|w_2{}^0\|_{0,\infty,K} \leqslant \left(1+\frac{1}{\nu}\right)^2 \operatorname*{Sup}_{\tilde{x}\in\tilde{K}} |\tilde{\alpha}\tilde{x}_1{}^2 + \tilde{\beta}\tilde{x}_2{}^2 + \tilde{\delta}\tilde{x}_1 + \tilde{\epsilon}\tilde{x}_2 + 1.$$

Using relations (2.3) of [1] which define γ_2, it is possible to estimate constants $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\delta}$, $\tilde{\epsilon}$ of (3.8), expressed in terms of the barycentric coordinates k_4, m_4 and n_4 of point a_4 with respect to the vertices of triangle $\alpha_1\,\alpha_2\,\alpha_3$ (relations (3.1) of [2]):

$$\tilde{\alpha} = -\frac{1}{2} - \frac{3}{2}\frac{m_4}{k_4 - n_4 - m_4}, \qquad \tilde{\beta} = -\frac{1}{2} - \frac{3}{2}\frac{n_4}{k_4 - m_4 - n_4}$$

$$\tilde{\delta} = \frac{1}{2} + 3\frac{m_4}{k_4 - n_4 - m_4}, \qquad \tilde{\epsilon} = \frac{1}{2} + 3\frac{n_4}{k_4 - m_4 - n_4}$$

with

$$k_4 - m_4 - n_4 = s(\hat{a}_2) \geqslant 1.$$

Also, $|\tilde{\alpha}|$, $|\tilde{\beta}|$, $|\tilde{\delta}|$ and $|\tilde{\epsilon}|$ are bounded if $|m_4|$ and $|n_4|$ are bounded. From the definition of m_4 and n_4, we deduce that

$$|m_4| = \frac{\delta(a_4,[\alpha_2,\alpha_3])}{\delta(\alpha_2,[\alpha_1,\alpha_3])} \quad\text{and}\quad |n_4| = \frac{\delta(a_4,[\alpha_1\alpha_2])}{\delta(\alpha_3,[\alpha_1\alpha_2])} \tag{3.10}$$

where $[\alpha_1, \alpha_3]$ (respectively $[\alpha_1, \alpha_2]$) denotes the straight line through points α_1 and α_3 (respectively through points α_1 and α_2).

From properties of the mapping F_K, we deduce that the images by F_K of the straight lines $[\alpha_1, \alpha_2]$ and $[\alpha_1, \alpha_3]$ are the straight lines $0\hat{x}$ and $0\hat{y}$. If l_{12} (respectively l_{13}) is an element of $P_1(\mathbf{R}^2)$ so that $l_{12}(x) = 0$ (respectively $l_{13}(x) = 0$) is an equation of the straight line $[\alpha_1, \alpha_2]$ (respectively of the straight line $[\alpha_1, \alpha_3]$ we have

$$\begin{aligned} |m_4| &= \left|\frac{l_{13}(a_4)}{l_{13}(\alpha_2)}\right| \leq \left|\frac{l_{13}(a_4)}{l_{13}(a_1)}\right| = \left|\frac{\hat{l}_{13}(\hat{a}_4)}{\hat{l}_{13}(\hat{a}_1)}\frac{s(\hat{a}_1)}{s(\hat{a}_4)}\right| \\ |m_4| &= \left|\frac{l_{12}(a_4)}{l_{12}(\alpha_3)}\right| \leq \left|\frac{l_{12}(a_4)}{l_{12}(a_3)}\right| = \left|\frac{\hat{l}_{12}(\hat{a}_4)}{\hat{l}_{12}(\hat{a}_3)}\frac{s(\hat{a}_4)}{s(\hat{a}_4)}\right| \end{aligned} \tag{3.11}$$

where $l_{13}{}^0F_K = \dfrac{\hat{l}_{13}}{s}$ and $l_{12}{}^0F_K = \dfrac{\hat{l}_{12}}{s}$ over $\hat{K}$. As

$$\hat{l}_{13}(\hat{a}_4) = \hat{l}_{13}(\hat{a}_1) \text{ and } \hat{l}_{12}(\hat{a}_4) = \hat{l}_{12}(\hat{a}_3),$$

we deduce, with proposition (3.4) of [2], that

$$|m_4| \leq 1 + \frac{1}{\nu} \text{ and } |n_4| \leq 1 + \frac{1}{\nu}. \tag{3.12}$$

Thus, $|\tilde{\alpha}|$, $|\tilde{\beta}|$, $|\tilde{\delta}|$ and $|\tilde{\epsilon}|$ are bounded by constants dependent on ν, and we obtain

$$\|w_2{}^0\|_{0,\infty,K} \leq \mathscr{C}_2(\nu).$$

We can show an analogous result for basis functions $w_1{}^0$, $w_3{}^0$ and $w_4{}^0$.

(2) *Basis functions* $w^1_{i,i+1}$ *and* $w^1_{i,i-1}$

We consider one of these functions, $w^1_{i,i+1}$ (the proof would be analogous for basis functions $w^1_{i,i-1}$).

With the relation (1.1), $w^1_{i,i+1}$ is defined, for each $x \in K$, by

$$w^1_{i,i+1}(x) = \frac{l(a_i)}{l^2_{i+2}(a_i)l_{i+3}(a_i)l_i(a_{i+1})}\,\frac{l^2_{i+2}(x)l_{i+3}(x)l_i(x)}{l(x)}$$

Then, from results on basis functions of first degree Serendip finite elements, it follows that

$$|w^1_{i,i+1}\|_{0,\infty,K} \leq \operatorname*{Sup}_{x\in K}\left|\frac{l_{i+2}(x)l_i(x)}{l_{i+2}(a_i)l_i(a_{i+1})}\right|.$$

Using properties of F_K we get

$$\frac{l_{i+2}(x)l_i(x)}{l_{i+2}(a_i)l_i(a_{i+1})} = \frac{\hat{l}_{i+2}(\hat{x})\hat{l}_i(\hat{x})}{\hat{l}_{i+2}(\hat{a}_i)\hat{l}_i(\hat{a}_{i+1})}\,\frac{s(\hat{a}_i)s(\hat{a}_{i+1})}{(s(\hat{x}))^2},$$

where $x = F_K(\hat{x})$.

Therefore, with propositions (3.2) and (3.4) of [2] and the geometry of $\hat{K}$,

$$|w^1_{i,i+1}\|_{0,\infty,K} \leq \left(1 + \frac{1}{\nu}\right)^2,$$

and the proof is completed.

Finally we obtain:

THEOREM 3.1.

Assume that the family of triangulations $(\tau_h)_{h\in\mathscr{H}}$ satisfies (3.1) and (3.2). Let $h \in \mathscr{H}$ and $K \in \tau_h$, let (K, P, Σ) be an Hermite rational "Wachspress-type" third degree finite element, and let Π be the P-interpolation operator with respect to Σ.

Then for each $m = 0, 1, 2, 3, 4$ there exists a constant $\mathscr{C}(m, 3, p, \nu, \sigma)$ independent of h and K such that, for each $v \in W^{4,p}(K)$,

$$|v - \pi v|_{m,p,K} \leq \mathscr{C}(m, 3, p, \nu, \sigma)\, [v]_{4,p,K} h_K^{4-m}. \tag{3.13}$$

Proof. The proof involves successive use of relations (2.6), (3.4), (3.5) and assumptions (3.2).

REMARK 3.1.

With Hermite rational third degree finite elements we can find the constants $\mathscr{C}(m, k, p, \nu, \sigma)$ of relations (3.13).

From proposition (3.1) if follows that there exist two constants $\mathscr{C}^0(\nu)$ and $\mathscr{C}^1(\nu)$ such that

$$\sum_{i\in I} \|w_i^0\|_{0,\infty,K} \leq \mathscr{C}^0(\nu)$$

$$\sum_{i\in I} (\|w_{i,i-1}^1\|_{0,\infty,K} + \|w_{i,i+1}^1\|_{0,\infty,K} \leq \mathscr{C}^1(\nu).$$

Then, if $\mathscr{C}(m, 4, \nu)$ is the constant introduced in lemma 3.1., as k is here equal to 3, the constant $\mathscr{C}(m, 3, p, \nu, \sigma)$ of theorem 3.1. can be written as

$$\mathscr{C}(m, 3, p, \nu, \sigma) = (m+1)^{1/p} \left[\frac{1}{6}\frac{\mathscr{C}^0(\nu)}{4-\frac{2}{p}} + \frac{1}{2}\frac{\mathscr{C}^1(\nu)}{3-\frac{2}{p}}\right] \sigma^m \mathscr{C}(m, 4, \nu)$$

when $0 \leq m \leq 3$ and

$$\mathscr{C}(4, 3, p, \nu, \sigma) = 5^{1/p}\left\{\frac{1}{6}\left[\frac{\mathscr{C}^0(\nu)}{4-\frac{2}{p}}(\sigma^4\mathscr{C}(4, 4, \nu) + \sigma^3\mathscr{C}(3, 4, \nu))\right]\right.$$
$$\left. + \frac{1}{2}\left[\frac{\mathscr{C}^1(\nu)}{3-\frac{2}{p}}(\sigma^4\mathscr{C}(4, 4, \sigma) + \sigma^3\mathscr{C}(3, 4, \nu))\right]\right\}$$

when $m = 4$. □

REMARK 3.2.

Theorem 3.2. shows that we get with Hermite rational "Wachspress-type" third degree finite elements the same asymptotic error estimates as for the usual finite elements of this degree.

REFERENCES

1. J. L. Gout, Construction of a Hermite rotational "Wachspress type" finite element. *Comput. Maths Applics* **5**, 337–347 (1979).
2. K. Apprato, R. Arcangeli et J. L. Gout, Sur les éléments finis rationnels de Wachspress, *Numer. Math.* **32**, 247–270 (1979).
3. E. L. Wachspress, *A Rotational Finite Element Basis*. Academic Press, New York (1975).
4. P. G. Ciarlet, *The Finite Element Method of Elliptic Problems*. North Holland, Amsterdam (1978).
5. P. G. Ciarlet et P. A. Raviart, General Lagrange and Hermite interpolation in $\mathbf{R}^n$ with applications in finite element methods. *Arch. Rat. Mech. Anal.* **46**, 177–199 (1972).
6. P. A. Raviart, Méthode des éléments finis, rédigé par J. M. Thomas. D.E.A. Analyse Numérique, Paris VI (1971–1972).
7. J. L. Gout, Estimation de l'erreur d'interpolation d'Hermite dans $\mathbf{R}^n$. *Numerische Mathematik* **28**, 407–429 (1977).
8. R. Wilhelmsem, A Markov inequality in several dimensions. *J. Appro. Theory* **11**, 216–220 (1974).
9. R. Arcangeli, J. L. Gout et R. Royer: Etude de l'erreur d'interpolation rationnelle de Wachspress sur un polygône. Publications Mathématiques de PAU (1976).

10. A. Adini and R. W. Clough, Analysis of plate bending by the finite element method, N.S.F. report G. 7337 (1961).
11. D. Apprato, Thèse 3ème cycle, Université de Pau, (1978).
12. R. Arcangeli et J. L. Gout, Sur l'évaluation de l'erreur d'interpolation de Lagrange dans un ouvert de $\mathbf{R}^n$, R.A.I.R.O. *Analyse Numérique*, **10**, 5–27 (1976).
13. Bernadou: Thèse d'Etat (1978).
14. P. G. Ciarlet, Numerical analysis of the finite element method. Université de Montréal (1975).
15. P. G. Ciarlet et P. A. Raviart, Interpolation theory over curved elements with applications to finite element methods. *Comp. Meth. Appl. Mech. Engng* **1**, 217–249 (1972).
16. P. G. Ciarlet et Wagschalc: Multipoint Taylor formulas and applications to Finite element method. *Numer. Math.* **17**, 84–100 (1971).
17. C. Coatmelec, Approximation et interpolation des fonctions différentiables de plusieurs variables. *Ann. Sc. Ecole Normale Sup* (3), **83**, 271–341 (1966).
18. J. Decloux, Méthode des éléments finis. Ecole polytechnique fédérale de Lausanne (1973).
19. J. L. Gout, Thése 3ème cycle (1976).
20. R. McLeod, Hermite Interpolation over curved finite elements. *J. Approx. Theory* **19**, 101–117 (1977).
21. J. Meinguet, Realistic estimates for generic constants in multivariable pointwise approximation. *Topics in Numerical Analysis II.* (Edited by J. J. H. Miller) Academic Press, New York (1975).
22. J. Necas, Les méthodes directes en théorie des équations elliptiques. Masson (1976).
23. E. L. Wachspress, A rational basis for function approximation. *Proc. Conf. of Appl. Numer. Anal. Dundee*, Springer-Verlag Lecture Notes in Mathematics, Vol. 228, pp. 223–252. Springer-Verlag, Berlin (1971).
24. E. L. Wachspress, A rational basis for function Approximation. Part. 2, curved sides. *J. Inst. Math. Appl.* **11**, 83–104 (1973).
25. E. L. Wachspress, Algebraic geometry foundations for finite element computation. *Conf. Numerical Sol. Diff. Equations, Dundee*, Springer-Verlag Lecture Notes in Mathematics, Vol. 363, pp. 177–188. Springer-Verlag, Berlin (1973).
26. O. C. Zienkiewicz, *La Méthode des Éléments Finis*. Ediscience, Paris (1973).