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CONSTRUCTION OF A HERMITE RATIONAL "WACHSPRESS TYPE" FINITE ELEMENT

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Abstract—This paper concerns the construction of a quadrilateral finite element whose interpolation space admits of rational fractions for basis functions of "Wachspress type"[1, 2]. The construction of this finite element, which is in a way the "rational" equivalent of the ADINI finite element[3, 4], is founded on a method analogous to the one used for Serendip degree-two finite element construction in[2]. The study of interpolation error is dealt with in a paper by Apprato, Arcangeli and Gout in this journal "Rational interpolation of Wachspress error estimates".

1. NOTATION AND REVIEW

1.1 *Geometrical elements* ([2])

Throughout this paper, K is a closed convex quadrilateral in $\mathbf{R}^2$, which is nondegenerate into a trapezium or parallelogram. The vertices a_i, $i \in I = Z/4Z$, of K are labelled so that a_i and a_{i+1} are consecutive and a_4 is the most distant vertex from the exterior diagonal d of K.

For all $i \in I$, d_i is the straight line passing through points a_{i-1} and a_i, and l_i is a $P_1(\mathbf{R}^2)$-element such that $l_i(x) = 0$ is the equation of d_i; α_1 denotes the intersection of the interior diagonals of K, α_2 (resp. α_3) the intersection of the straight lines d_1 and d_3 (resp. d_2 and d_4).

Finally, l defines a $P_1(\mathbf{R}^2)$ element such that $l(x) = 0$ is the equation of d.

Moreover, we denote by $\hat{K}$ the usual reference square whose vertices are: $\hat{a}_1 = (-1, 1)$, $\hat{a}_2 = (-1, -1)$, $\hat{a}_3 = (1, -1)$ and $\hat{a}_4 = (1, 1)$.

We let $\hat{d}_i$ be the straight line passing through points $\hat{a}_{i-1}$ and $\hat{a}_i (i \in I)$ and $\hat{l}_i$ be a $P_1(\mathbf{R}^2)$ element such that $\hat{l}_i(\hat{x}) = 0$ is the equation of $\hat{d}_i$.

1.2. *Application to* F_K

We recall the definition given in the paragraph 3 of [2] as well as some properties which are proved in the same paragraph. Let (k_4, m_4, n_4) be the barycentric coordinates of a_4 with respect to the points α_1, α_2, and α_3. We define $s \in P_1(\mathbf{R}^2)$ by:

$$s(\hat{x}_1, \hat{x}_2) = k_4 + m_4\hat{x}_1 + n_4\hat{x}_2.$$

We now consider the mapping F_K of each point $(\hat{x}_1, \hat{x}_2)$ of $\mathbf{R}^2$ such that $s(\hat{x}_1, \hat{x}_2) \neq 0$ to the point (x_1, x_2) of $\mathbf{R}^2$:

$$(x_1, x_2) = F_K(\hat{x}_1, \hat{x}_2)$$

with

$$x_1 = \frac{\alpha_{1,1}k_4 + \alpha_{1,2}m_4\hat{x}_1 + \alpha_{1,3}n_4\hat{x}_2}{s(\hat{x}_1, \hat{x}_2)}$$

$$x_2 = \frac{\alpha_{2,1}k_4 + \alpha_{2,2}m_4\hat{x}_1 + \alpha_{2,3}n_4\hat{x}_2}{s(\hat{x}_1, \hat{x}_2)}, \tag{1.1}$$

where $\alpha_i = (\alpha_{i,1}, \alpha_{i,2})$ for $i = 1, 2, 3$.

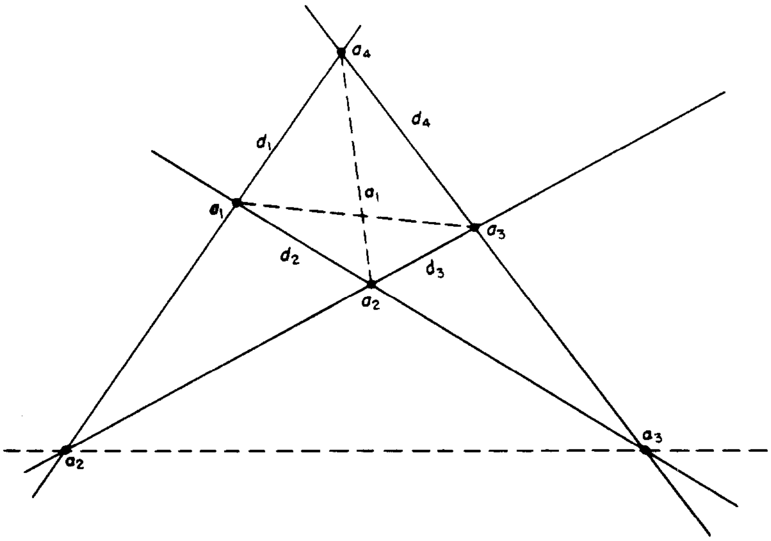

Fig. 1.

In what follows, we will be using some properties of F_K proved in proposition (3.1) of [2], and particularly:

$$F_K(0) = \alpha_1,\ F_K(\hat{a}_i) = a_i \text{ for all } i \in I$$

The image, by F_K, of a straight line different from the straight line S whose equation is $s(\hat{x}) = 0$ (eventually without its intersection with this line) is a straight line (eventually without its intersection with the line d). (1.2)

$$F_K \text{ is a } C^\infty\text{-diffeomorphism of } \hat{K} \text{ on } K.$$

Finally we recall proposition (3.2) of [2]:

$$k_4 > 1,\quad m_4 < 0,\quad n_4 < 0$$

and

$$\forall \hat{x} \in \hat{K},\quad s(\hat{x}) = \frac{l(a_4)}{l(x)} \text{ with } x = F_K(\hat{x}) \text{ and } s(\hat{x}) \geq 1, \tag{1.3}$$

as well as proposition (3.3) of [2]: For each $k \in \mathbf{N}$

$$\{\hat{v}\colon \hat{K} \mapsto \mathbf{R}, \exists v \in P_k(K), \hat{v} = voF_K\} = \left\{\hat{v}\colon \hat{K} \to \mathbf{R},\ \exists \hat{p} \in P_k(\hat{K}),\ \hat{v} = \frac{\hat{p}}{s^k}\right\}$$

and

$$\{v\colon K \mapsto \mathbf{R},\ \exists \hat{v} \in P_k(\hat{K}),\ \hat{v} = \hat{v}oF_K^{-1}\} = \left\{v\colon K \mapsto \mathbf{R},\ \exists p \in P_k(K),\ v = \frac{p}{l^k}\right\} \tag{1.4}$$

where F_K, s and l replace $F_{K|\hat{K}}$, $s|_{\hat{K}}$ and $l|_{\hat{K}}$.

1.3 *Finite element*

For other notation and definitions on finite elements [2, 4, 5]. We recall that a finite element (K, P_K, Σ_K) is of degree k if the interpolation space P_K satisfies

$$P_K \supset P_k(K) \text{ and } P_K \not\supset P_{k+1}(K). \tag{1.5}$$

See also [4] and [5] for the definition of finite element of class $C^p (p \in \mathbf{N})$.

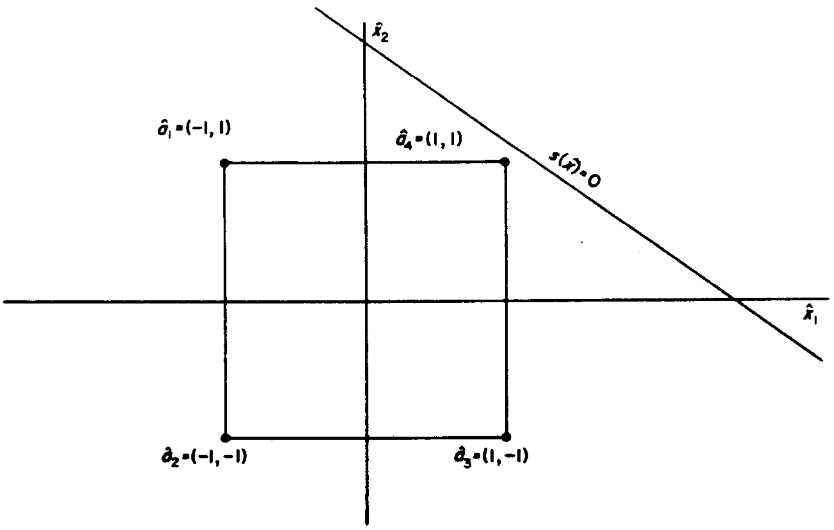

Fig. 2.

Finally, as usual, for each subset A of $\mathbf{R}^2$, $P_k(A)$ ($k \in \mathrm{N}$) is the A-restriction of the vectorial space of the polynomial functions of degree $\leqslant k$ in two variables.

2. PRELIMINARY RESULTS

Let K be a convex quadrilateral defined as in the first paragraph, and, for some $i \in I$, let γ_i be a $P_2(\mathbf{R}^2)$-element so that $\gamma_i(x) = 0$ is the equation of a conic Γ_i with

$$\begin{gathered}\gamma_i(a_{i+1}) = \gamma_i(a_{i-1}) = 0\\ D(l_{i+2}\gamma_i)(a_i)\cdot(a_{i+1}-a_i) = 0\\ D(l_{i+3}\gamma_i)(a_i)\cdot(a_{i-1}-a_i) = 0\\ \hat{\gamma}_i(\hat{x}),\text{ where } \hat{\gamma}_i = s^2(\gamma_i o F_K),\text{ has no terms } \hat{x}_1\hat{x}_2.\end{gathered} \tag{2.1}$$

Remark 2.1

Given the properties of F_K (cf. relations (1.4)), $\hat{\gamma}_i$ is a $P_2(\mathbf{R}^2)$ element so that $\hat{\gamma}_i(\hat{x}) = 0$ is an equation of the conic $\hat{\Gamma}_i$, which is the image of Γ_i by F_K^{-1}. □

We deduce from properties of F_K that, for each $j \in I$ and for each $i \in I$ the relation

$$D(l_j\gamma_i)(a_i)\cdot(a_{i+1}-a_i) = 0$$

is equivalent to

$$D\left(\frac{\hat{l}_j\hat{\gamma}_i}{s^3}\right)(\hat{a}_i)\cdot(\hat{a}_{i+1}-\hat{a}_i) = 0 \tag{2.2}$$

because, from (1.4) we can say that for each $j \in I$

$$l_j = c_j\frac{\hat{l}_j}{s}oF_K^{-1}$$

where c_j is an appropriate constant, and consequently, from (2.1):

$$\begin{aligned}D(l_j\gamma_i)(a_i)\cdot(a_{i+1}-a_i) &= D\left(\frac{\hat{l}_j\hat{\gamma}_i}{s^3}oF_K^{-1}\right)(F_K(\hat{a}_i)).\ [F_K(\hat{a}_{i+1})-F_K(\hat{a}_i)]\\ &= D\left(\frac{\hat{l}_j\hat{\gamma}_i}{s^3}\right)(\hat{a}_i)oDF_K^{-1}(F_K(\hat{a}_i))\cdot(F_K(\hat{a}_{i+1})-F_K(\hat{a}_i)).\end{aligned}$$

Then we deduce the relation (2.2).
Now we have

PROPOSITION 2.1

For each $i \in I$, the conic Γ_i which satisfies properties (2.1) exists and is unique.

Proof. Using (2.2) and the properties of the mapping F_K, proof of this proposition is equivalent to proving that, for any $i \in I$, there exists a unique conic $\hat{\Gamma}_i$ (remark 2.1), or that there exists a unique element $\hat{\gamma}_i \in P_2(\mathbf{R})$ such that:

$$\hat{\gamma}_i(\hat{a}_{i+1}) = \hat{\gamma}_i(\hat{a}_i) = 0$$

$$D\left(\frac{\hat{l}_{i+2}\hat{\gamma}_i}{s^3}\right)(\hat{a}_i)\cdot(\hat{a}_{i+1}-\hat{a}_i) = 0$$

$$D\left(\frac{\hat{l}_{i+3}\hat{\gamma}_i}{s^3}\right)(\hat{a}_i)\cdot(\hat{a}_{i-1}-\hat{a}_i) = 0$$

$$\hat{\gamma}_i(\hat{x}_1, \hat{x}_2) = \alpha_i\hat{x}_1{}^2 + \beta_i\hat{x}_2{}^2 + \delta_i\hat{x}_1 + \epsilon_i\hat{x}_2 + 1, \tag{2.3}$$

where α_i, β_i, δ_i and $\epsilon_i \in \mathbf{R}$, and where it is assumed that $\hat{\Gamma}_i$ does not contain the point of origin (Γ_i does not contain point a_1).
On the other hand, if we note

$$\lambda = \frac{m_4}{k_4}, \qquad \mu = \frac{n_4}{k_4}, \tag{2.4}$$

the equation of the straight line S is

$$\sigma(\hat{x}_1, \hat{x}_2) = k_4 s(\hat{x}_1, \hat{x}_2) = \lambda\hat{x}_1 + \mu\hat{x}_2 + 1 = 0. \tag{2.5}$$

Application of conditions (2.3) to points $\hat{a}_1, \hat{a}_2, \hat{a}_3$ and $\hat{a}_4$ with, for convenience, $\alpha, \beta, \delta, \epsilon$ instead of α_i, β_i, δ_i and ξ_i shows that we successively get the linear systems for the conics Γ_i:

Conic $\hat{\Gamma}_1$

$$\alpha + \beta - \delta + \epsilon = -1$$

$$\alpha + \beta + \delta + \epsilon = -1$$

$$\begin{gathered}\left(1-\frac{6\mu}{\sigma(\hat{a}_1)}\right)\alpha + \left(5-\frac{6\mu}{\sigma(\hat{a}_1)}\right)\beta + \left(-1+\frac{6\mu}{\sigma(\hat{a}_1)}\right)\delta + \left(3-\frac{6\mu}{\sigma(\hat{a}_1)}\right)\epsilon = -1+\frac{6\mu}{\sigma(\hat{a}_1)}\\ \left(5+\frac{6\lambda}{\sigma(\hat{a}_1)}\right)\alpha + \left(1+\frac{6\lambda}{\sigma(\hat{a}_1)}\right)\beta + \left(-3-\frac{6\lambda}{\sigma(\hat{a}_1)}\right)\delta + \left(1+\frac{6\lambda}{\sigma(\hat{a}_1)}\right)\epsilon = -1-\frac{6\lambda}{\sigma(\hat{a}_1)}.\end{gathered} \tag{2.6}$$

The determinant $\hat{\Delta}_1$ of this system is equal to $32[(\lambda - \mu + 2)/(\sigma(\hat{a}_1))]$, and $\hat{\Delta}_1 \neq 0$ because if $\lambda = \mu - 2$, the straight line S belongs to the group of straight lines which have equations $(\mu - 2)\hat{x}_1 + \mu\hat{x}_2 + 1 = 0$ and thus contain the point $[(1/2), (-1/2)]$, which is impossible since $s(\hat{x}) \neq 0$ for $\hat{x} \in \hat{K}$ (cf. (2.5)).

Conic $\hat{\Gamma}_2$

$$\alpha + \beta + \delta - \epsilon = -1$$

$$\alpha + \beta - \delta + \epsilon = -1$$

$$\begin{gathered}\left(5+\frac{6\lambda}{\sigma(\hat{a}_1)}\right)\alpha + \left(1+\frac{6\lambda}{\sigma(\hat{a}_2)}\right)\beta + \left(-3-\frac{6\lambda}{\sigma(\hat{a}_2)}\right)\delta + \left(-1-\frac{6\lambda}{\sigma(\hat{a}_2)}\right)\epsilon = -1-\frac{6\lambda}{\sigma(\hat{a}_2)}\\ \left(1+\frac{6\mu}{\sigma(\hat{a}_2)}\right)\alpha + \left(5+\frac{6\mu}{\sigma(\hat{a}_2)}\right)\beta + \left(-1-\frac{6\mu}{\sigma(\hat{a}_2)}\right)\delta + \left(-3-\frac{6\mu}{\sigma(\hat{a}_2)}\right)\epsilon = -1-\frac{6\mu}{\sigma(\hat{a}_2)}.\end{gathered} \tag{2.7}$$

The determinant $\hat{\Delta}_2$ of this system is equal to $32[(2 + \lambda + \mu)/(\sigma(\hat{a}_2))]$ and $\hat{\Delta}_2 \neq 0$ because, if

$\lambda + \mu = -2$, the straight line S belongs to the group of straight lines which have equations $(-\mu - 2)\hat{x}_1 + \mu\hat{x}_2 + 1 = 0$ and contain the point $[(1/2), (1/2)]$, which is impossible since $s(\hat{x}) \neq 0$ for $\hat{x} \in \hat{K}$ (*cf.* 2.5).

Conic $\hat{\Gamma}_3$

$$\begin{gathered}
\alpha + \beta + \delta + \epsilon = -1 \\
\alpha + \beta - \delta - \epsilon = -1 \\
\left(1 + \frac{6\mu}{\sigma(\hat{a}_3)}\right)\alpha + \left(5 + \frac{6\mu}{\sigma(\hat{a}_3)}\right)\beta + \left(1 + \frac{6\mu}{\sigma(\hat{a}_3)}\right)\delta + \left(-3 - \frac{6\mu}{\sigma(\hat{a}_3)}\right)\epsilon = -1 - \frac{6\mu}{\sigma(\hat{a}_3)} \\
\left(5 - \frac{6\lambda}{\sigma(\hat{a}_3)}\right)\alpha + \left(1 - \frac{6\lambda}{\sigma(\hat{a}_3)}\right)\beta + \left(3 - \frac{6\lambda}{\sigma(\hat{a}_3)}\right)\delta + \left(-1 + \frac{6\lambda}{\sigma(\hat{a}_3)}\right)\epsilon = -1 + \frac{6\lambda}{\sigma(\hat{a}_3)}.
\end{gathered} \tag{2.8}$$

The determinant $\hat{\Delta}_3$ of this system is equal to $32[(\lambda - \mu - 2)/(\sigma(\hat{a}_3))]$ and $\hat{\Delta} \neq 0$ because, if $\lambda = \mu + 2$, the straight line S belongs to the group of straight lines which have equations $(\mu + 2)\hat{x}_1 + \mu\hat{x}_2 + 1 = 0$ and contain the point $[-(1/2), (1/2)]$, which is impossible since $s(\hat{x}) \neq 0$ for $\hat{x} \in \hat{K}$ (*cf.* 2.5).

Conic $\hat{\Gamma}_4$

$$\begin{gathered}
\alpha + \beta - \delta + \epsilon = -1 \\
\alpha + \beta + \delta - \epsilon = -1 \\
\left(5 - \frac{6\lambda}{\sigma(\hat{a}_4)}\right)\alpha + \left(1 - \frac{6\lambda}{\sigma(\hat{a}_4)}\right)\beta + \left(3 - \frac{6\lambda}{\sigma(\hat{a}_4)}\right)\delta + \left(1 - \frac{6\lambda}{\sigma(\hat{a}_4)}\right)\epsilon = -1 + \frac{6\lambda}{\sigma(\hat{a}_4)} \\
\left(1 - \frac{6\mu}{\sigma(\hat{a}_4)}\right)\alpha + \left(5 - \frac{6\mu}{\sigma(\hat{a}_4)}\right)\beta + \left(1 - \frac{6\mu}{\sigma(\hat{a}_4)}\right)\delta + \left(3 - \frac{6\mu}{\sigma(\hat{a}_4)}\right)\epsilon = -1 - \frac{6\mu}{\sigma(\hat{a}_4)}.
\end{gathered} \tag{2.9}$$

The determinant $\hat{\Delta}_4$ of this system is equal to $32[(2 - \lambda - \mu)/(\sigma(\hat{a}_4))]$ and $\hat{\Delta}_4 \neq 0$ because, if $\lambda = -\mu + 2$, the straight line S belongs to the group of straight lines which have equations $(2 - \mu)\hat{x}_1 + \mu\hat{x}_2 + 1 = 0$ and contain the point $[-(1/2), -(1/2)]$, which is impossible since $s(\hat{x}) \neq 0$ for $\hat{x} \in \hat{K}$ (*cf.* 2.5).

Thus we deduce that, for each $i \in I$, a unique conic $\hat{\Gamma}_i$, which satisfies properties (2–3), corresponds to any point $\hat{a}_i$ and the proof is complete.

It is necessary to study separately the four vertices of the quadrilateral which, at first sight, do not have a symmetrical position in the quadrilateral K (Fig. 1).

3. FINITE ELEMENT CONSTRUCTION

Let K be a convex quadrilateral defined as in paragraph 1.

3.1 *Interpolation space* $\mathrm{P_K}$ *and set* Σ_K

With the notation of paragraph 1, we define functions w_i^0 $(i \in I)$, $w_{i,j}^1$ $(i \in I,\ j = i - 1,\ i + 1)$ where, for each $x \in K$:

$$\begin{aligned}
w_i^0(x) &= \frac{l(a_i)}{l_{i+2}(a_i)l_{i+3}(a_i)\gamma_i(a_i)}\,\frac{l_{i+2}(x)l_{i+3}(x)\gamma_i(x)}{l(x)} \\
w_{i,i-1}^1(x) &= \frac{l(a_i)}{l_{i+1}(a_{i-1})l_{i+2}(a_i)l_{i+3}^2(a_i)}\,\frac{l_{i+1}(x)l_{i+2}(x)l_{i+3}^2(x)}{l(x)} \\
w_{i,i+1}^1(x) &= \frac{l(a_i)}{l_{i+2}^2(a_i)l_{i+3}(a_i)l_i(a_{i+1})}\,\frac{l_{i+2}^2(x)l_{i+3}(x)l_i(x)}{l(x)}
\end{aligned} \tag{3.1}$$

where, for each $i \in I$, γ_i is an element of $P_2(\mathbf{R}^2)$ which satisfies relations (2.1).

REMARK 3.1

From relations (2.1), one can deduce that the conic Γ_i, whose equation is $\gamma_i(x) = 0$, contains the points a_{i+1} and a_{i-1}, but neither α_i, nor the point a_i. The conic Γ_i is completely defined by five points: a_{i+1}, a_{i-1} and three other points which may be chosen from the images by F_K of three points (distinct from $\hat{a}_{i+1}$ and $\hat{a}_{i-1}$) of the conic $\hat{\Gamma}_i$ which satisfies relations (2.3).

Definition 3.1

Let P_K be the vector space generated by functions $w_i^0 (i \in I)$ and $w_{i,j}^1$ $(i \in I, j = i-1, i+1)$ (see relations 3.1).

We introduce:

Definition 3.2

Let Σ_K be the set of linearly independent linear forms defined over the space P_K by

$$\Sigma_K = \{v \to v(a_i),\ i \in I;\ v \to Dv(a_i) \cdot (a_j - a_i),\ i \in I,\ j = i-1,\ i+1\}. \tag{3.2}$$

3.2 P_K-*unisolvence of* Σ_K

PROPOSITION 3.1

Let P_K and Σ_K be defined by (3.1) and (3.2), then (K, P_K, Σ_K) is a finite element.

Proof. First,

$$\dim P_K = \dim \Sigma_K = 12$$

Thus, it suffices to prove that the set Σ_K is P_K-unisolvent. As a matter of fact, we shall prove that functions w_i^0, $w_{i,i-1}^1$ and $w_{i,i+1}^1$ are the basis functions of P_K with reference to Σ_K.

Functions w_i^0

It is easy to show that

$$\forall i \in I,\ \forall j \in I,\ w_i^0(a_j) = \delta_{ij}. \tag{3.3}$$

Also, because w_i^0 vanishes on d_{i+2} and d_{i+3}, and γ_i on a_{i+1} and a_{i-1}, we obtain

$$\forall i \in I,\ \forall j \in I,\ j \neq i,\ Dw_i^0(a_j) = 0. \tag{3.4}$$

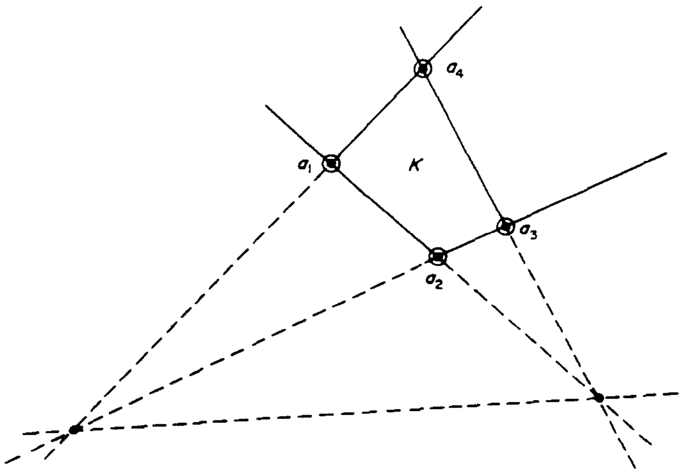

Fig. 3.

Moreover,

$$Dw_i^0(a_i)\cdot(a_{i+1}-a_i)=\lim_{\alpha\to 0}\frac{w_i^0[a_i+\alpha(a_{i+1}-a_i)]-w_i^0(a_i)}{\alpha}$$

$$=\lim_{\alpha\to 0}\frac{1}{\alpha}\left[\frac{l_{i+2}[a_i+\alpha(a_{i+1}-a_i)]l_{i+3}[a_i+\alpha(a_{i+1}-a_i)]\gamma_i[a_i+\alpha(a_{i+1}-a_i)]l(a_i)}{l_{i+2}(a_i)l_{i+3}(a_i)\gamma_i(a_i)l[a_i+\alpha(a_{i+1}-a_i)]}-1\right]$$

which, since l_{i+3}/l is constant on the segment $[a_i, a_{i+1}]$, is equal to

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\left[\frac{l_{i+2}[a_i+\alpha(a_{i+1}-a_i)]\gamma_i[a_i+\alpha(a_{i+1}-a_i)]}{l_{i+2}(a_i)\gamma_i(a_i)}-1\right]$$

or

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\,\frac{(l_{i+2}\gamma_i)[a_i+\alpha(a_{i+1}-a_i)]-(l_{i+2}\gamma_i)(a_i)}{l_{i+2}(a_i)\gamma_i(a_i)}$$

$$=\frac{D(l_{i+2}\gamma_i)(a_i)\cdot(a_{i+1}-a_i)}{l_{i+2}(a_i)\gamma_i(a_i)}.$$

Therefore, as γ_i $(i\in I$ satisfies conditions (2.1), we have

$$Dw_i^0(a_i)\cdot(a_{i+1}-a_i)=0$$

and we can also show that

$$Dw_i^0(a_i)\cdot(a_{i-1}-a_i)=0.$$

Then

$$\forall i\in I\quad Dw_i^0(a_i)=0. \tag{3.5}$$

Using relations (3.3), (3.4) and (3.5), we obtain

$$\forall i\in I,\ \forall j\in I\ \ w_i^0(a_j)\ =\delta_{ij}\ \text{and}\ Dw_i^0(a_j)=0. \tag{3.6}$$

Functions $w^1_{i,i+1}$ *and* $w^1_{i,i-1}$

Now consider one of these functions: $w^1_{i,i+1}$. We have

$$\forall i\in I,\ \forall j\in I,\ \ w^1_{i,i+1}(a_j)=0 \tag{3.7}$$

and, using the definition of $w^1_{i,i+1}$, we get

$$Dw^1_{i,i+1}(a_i)\cdot(a_{i+1}-a_i)=1. \tag{3.8}$$

so that

$$\forall i\in I,\ \forall j\in I,\ \ j\neq i\ \ Dw^1_{i,i+1}(a_j)=0. \tag{3.9}$$

Finally,

$$Dw^1_{i,i+1}(a_i)\cdot(a_{i+3}-a_i)=\lim_{\alpha\to 0}\frac{w^1_{i,i+1}[a_i+\alpha(a_{i+3}-a_i)]-w^1_{i,i+1}(a_i)}{\alpha}=0, \tag{3.10}$$

because $w^1_{i,i+1}(a_i)=0$ as a result of (3.7), and $w^1_{i,i+1}[a_i+\alpha(a_{i+3}-a_i)]=0$ because $l_i|a_i+\alpha(a_{i+3}-a_i)|=0$.

We can obtain analogous results with the functions $w^1_{i,i-1}$. Thus, with relations (3.7), (3.8), (3.9) and (3.10) we deduce the properties:

$$\begin{aligned}
&\forall i \in I,\ \forall j \in I,\ w^1_{i,i+1}(a_j) = 0, \quad w^1_{i,i-1}(a_j) = 0\\
&\forall i \in I,\ \forall j \in I,\ j \neq i\ Dw^1_{i,i+1}(a_j) = 0, \quad Dw^1_{i,i-1}(a_j) = 0,\\
&\forall i \in I,\ Dw^1_{i,i+1}(a_i)\cdot(a_{i+1} - a_i) = 1, \quad Dw^1_{i,i+1}(a_i)\cdot(a_{i-1} - a_i) = 0,\\
&\forall i \in I,\ Dw^1_{i,i-1}(a_i)\cdot(a_{i+1} - a_i) = 0, \quad Dw^1_{i,i-1}(a_i)\cdot(a_{i-1} - a_i) = 1.
\end{aligned} \tag{3.11}$$

The relations (3.6) and (3.11) demonstrate that the functions $w_i^0 (i \in I)$, $w^1_{i,i+1}$ and $w^1_{i,i-1}$ are basis functions of P_K with respect to Σ_K and the proof is complete.

REMARK 3.2

We note the analogy with the ADINI finite element [3, 4] which is defined on a rectangle and which has polynomial basis functions.

4. PROPERTIES OF THE FINITE ELEMENT

4.1 *Study of the degree of the finite element*

PROPOSITION 4.1

The finite element (K, P_K, Σ_K) defined by (3.1) and (3.2) is 3rd-degree.

Proof. Clearly, we have $P_K \not\supset P_4(K)$ because $\dim P_K = 12$ and $\dim P_4(K) = 15$. Let Ψ be an element of $P_3(K)$ and consider the function v defined over K by

$$v = \Psi - \Big\{\sum_{i\in I} \Psi(a_i) w_i^0 + \sum_{\substack{i\in I\\ j=i-1,i+1}} [D\Psi(a_i)\cdot(a_j - a_i)] w^1_{i,j}\Big\}. \tag{4.1}$$

Applying proposition (3.3) of [2], we get

$$\Psi o F_K = \frac{\hat\Psi}{s^3} \text{ with } \hat\Psi \in P_3(\hat K)$$

$$\Psi(a_i)(w_i^0 o F_K) = \frac{\hat\Psi(\hat a_i)\hat w_i^0}{s^3} \text{ with } \hat w_i^0(\hat x) = \frac{\hat l_{i+2}(\hat x)\hat l_{i+3}(\hat x)\hat\gamma_i(\hat x)}{\hat l_{i+2}(\hat a_i)\hat l_{i+3}(\hat a_i)\hat\gamma_i(\hat a_i)}$$

$$[D\Psi(a_i)\cdot(a_j - a_i)](w^1_{i,j} o F_K) = D\Big(\frac{\hat\Psi}{s^3}\Big)(\hat a_i) o DF_K^{-1}(F_K(\hat a_i))\cdot(F_K(\hat a_j) - F_K(\hat a_i))\frac{\hat w^1_{i,j}}{s^3}, \tag{4.2}$$

where, for each $i \in I$,

$$\hat w^1_{i,i+1}(\hat x) = s^2(\hat a_i)s(\hat a_{i+1})\frac{\hat l^2_{i+2}(\hat x)\hat l_{i+3}(\hat x)\hat l_i(\hat x)}{\hat l^2_{i+2}(\hat a_i)\hat l_{i+3}(\hat a_i)\hat l_i(\hat a_{i+1})}$$

$$w^1_{i,i-1}(x) = s^2(\hat a_i)s(\hat a_{i-1})\frac{\hat l_{i+1}(\hat x)\hat l_{i+2}(\hat x)\hat l^2_{i+3}(\hat x)}{\hat l_{i+1}(\hat a_{i-1})\hat l_{i+2}(\hat a_i)\hat l^2_{i+3}(\hat a_i)}.$$

Then, $v o F_K$ can be expressed as $\hat v/s^3$, where, considering (4.2), $\hat v$ is defined by:

$$\hat v = \hat\Psi - \sum_{i\in I}\hat\Psi(\hat a_i)\hat w_i^0 - \sum_{\substack{i\in I\\ j=i-1,i+1}} \times D\Big(\frac{\hat\Psi}{s^3}\Big)(\hat a_i) o DF_K^{-1}(F_K(\hat a_i))\cdot(F_K(\hat a_j) - F_K(\hat a_i))\hat w^1_{i,j} \tag{4.3}$$

The expressions of the functions $\hat w_i^0$ and $\hat w^1_{i,j}$, as well as the choice of $\hat\Psi$ prove that $\hat v \in P_4(\hat K)$.

Let us consider now the restriction $\hat v'$ of $\hat v$ to any side K' of $\hat K$: $\hat v'$ is a polynomial function.

Consequently, from expressions $\hat{w}_i^0$ and $\hat{w}_{i,j}^1$ indicated in (4.2) and using geometry of $\hat{K}$, we deduce that

$$\hat{v}' = v|_{K'} \in P_3(K').$$

Thus, if, for example, K' is the segment $[\hat{a}_l, \hat{a}_{l+1}]$ it follows from relations (4.2) and (4.3) that

$$\hat{v}'(\hat{a}_l) = \hat{v}'(\hat{a}_{l+1}) = 0. \tag{4.4}$$

On the other hand, let us consider now, for example, the expression

$$D\left(\frac{\hat{v}}{s^3}\right)(\hat{a}_l) \cdot (\hat{a}_{l+1} - \hat{a}_l).$$

From the definition of the functions $\hat{w}_i^0$ $(i \in I)$ and relations (2.3) we can show that

$$\forall i \in I \;\; D\left(\frac{\hat{w}_i^0}{s^3}\right)(\hat{a}_l) \cdot (\hat{a}_{l+1} - \hat{a}_l) = 0. \tag{4.5}$$

From the definition of the functions $\hat{w}_{i,j}^1$ and relations (3.12) we get

$$\begin{aligned}\sum_{\substack{i\in I\\ j=i-1,i+1}} & D\left(\frac{\hat{\Psi}}{s^3}\right)(\hat{a}_i) o DF_K^{-1}(F_K(\hat{a}_i))(F_K(\hat{a}_j) - F_K(\hat{a}_i)) D\left(\frac{\hat{w}_{i,j}}{s^3}\right)(\hat{a}_l) \cdot (\hat{a}_{l+1} - \hat{a}_l)\\ &= D\left(\frac{\hat{\Psi}}{s^3}\right)(\hat{a}_l) o DF_K^{-1}(\hat{a}_l)(a_{l+1} - a_l) D\left(\frac{\hat{w}_{l,l+1}}{s^3}\right)(\hat{a}_l) \cdot (\hat{a}_{l+1} - \hat{a}_l).\end{aligned}$$

But we have

$$D\left(\frac{\hat{w}_{l,l+1}^1}{s^3}\right)(\hat{a}_l) \cdot (\hat{a}_{l+1} - \hat{a}_l) = Dw_{l,l+1}^1(a_l) o DF_K(\hat{a}_l) \cdot (\hat{a}_{l+1} - \hat{a}_l)$$

and also, with the properties of F_K

$$DF_K(\hat{a}_l) \cdot (\hat{a}_{l+1} - \hat{a}_l) = \lambda(a_{l+1} - a_l), \quad \lambda \in \mathbf{R}^*.$$

Then, using

$$DF_K^{-1}(a_l) \cdot (a_{l+1} - a_l) = (DF_K(\hat{a}_l))^{-1}(a_{l+1} - a_l) = \frac{\hat{a}_{l+1} - \hat{a}_l}{\lambda},$$

we obtain, finally:

$$\begin{aligned}\sum_{\substack{i\in I\\ j=i-1,i+1}} & D\left(\frac{\hat{\Psi}}{s^3}\right)(\hat{a}_i) o DF_K^{-1}(F_K(\hat{a}_i))(F_K(\hat{a}_j) - F_K(\hat{a}_i)) D\left(\frac{\hat{w}_{i,j}^1}{s^3}\right)(\hat{a}_l) \cdot (\hat{a}_{l+1} - \hat{a}_l)\\ &= D\left(\frac{\hat{\Psi}}{s^3}\right)(\hat{a}_l) \cdot (\hat{a}_{l+1} - \hat{a}_l).\end{aligned} \tag{4.6}$$

From (4.5) and (4.6) we deduce

$$D\left(\frac{\hat{v}}{s^3}\right)(\hat{a}_l) \cdot (\hat{a}_{l+1} - \hat{a}_l) = 0,$$

and with $\hat{v}(\hat{a}_l) = 0$, we get

$$D\hat{v}'(\hat{a}_l) \cdot (\hat{a}_{l+1} - \hat{a}_l) = 0. \tag{4.7}$$

Likewise we can show that

$$D\hat{v}'(\hat{a}_{l+1}) \cdot (\hat{a}_l - \hat{a}_{l+1}) = 0. \tag{4.8}$$

Since $\hat{v}' \in P_3(K')$ satisfies four vanishing and independent conditions, given by (4.4), (4.7) and (4.8), it follows that

$$\hat{v}' = 0 \quad \text{over } K'$$

Thus, $\hat{v}$ vanishes on the four sides of the square of reference and we can write

$$\hat{v}(\hat{x}_1, \hat{x}_2) = \hat{c}(1-\hat{x}_1^2)(1-\hat{x}_2^2), \quad \hat{c} \in \mathbf{R}.$$

Identifying terms $\hat{x}_1^2\hat{x}_2^2$ in (4.3), we show (since the $\hat{\gamma}_i$'s have no terms $\hat{x}_1\hat{x}_2$) that

$$\hat{c} = 0,$$

and therefore that

$$v = 0,$$

so that, with (4.1),

$$\Psi \in P_K$$

as was to be proved.

PROPOSITION 4.2

The finite element (K, P_K, Σ) defined by (3.1) and (3.2) is of class C^0.

Proof. Let Ω be an open bounded subset polygon in $\mathbf{R}^2$ and let τ_h be a "triangulation" of $\bar{\Omega}$ by quadrilaterals K defined as in paragraph 1. Let V_h be a finite element space whose generic element (K, P_K, Σ_K) is given by (3.1) and (3.2).

Let us consider now $v \in V_h$ and two adjacent quadrilaterals K_j and K_l with common side $K' = [a_i, a_{i+1}]$. It is known that

$$v|_{K_j} = p_j \in P(K_j), \quad v|_{K_l} = p_l \in P(K_l).$$

On the other hand, from formulas (3.1), we can say that the restriction to K of the basis functions are elements of $P_3(K')$ (properties of the group of straight lines).

Also we get

$$(p_j - p_l)|_{K'} \in P_3(K') \tag{4.9}$$

and further,

$$\begin{aligned} (p_j - p_l)(a_i) = (p_j - p_l)(a_{i+1}) &= 0 \\ D(p_j - p_l)(a_i) \cdot (a_{i+1} - a_i) &= 0 \\ D(p_j - p_l)(a_{i+1}) \cdot (a_i - a_{i+1}) &= 0. \end{aligned} \tag{4.10}$$

From relations (4.9) and (4.10) we deduce that

$$(p_j - p_l)|_{K'} = 0,$$

this being the desired result.

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