Homogenization of second order discrete model with local perturbation and application to traffic flow

N. Forcadel[1], W. Salazar[1], M. Zaydan[1]

May 4, 2016

Abstract

The goal of this paper is to derive a traffic flow macroscopic model from a second order microscopic model with a local perturbation. At the microscopic scale, we consider a Bando model of the type following the leader, i.e the acceleration of each vehicle depends on the distance of the vehicle in front of it. We consider also a local perturbation like an accident at the roadside that slows down the vehicles. After rescaling, we prove that the "cumulative distribution functions" of the vehicles converges towards the solution of a macroscopic homogenized Hamilton-Jacobi equation with a flux limiting condition at junction which can be seen as a LWR (Lighthill-Whitham-Richards) model.

AMS Classification: 35D40, 90B20, 35B27, 35F20, 45K05.

Keywords: specified homogenization, Hamilton-Jacobi equations, integro-differential operators, Slepčev formulation, viscosity solutions, traffic flow, microscopic models, macroscopic models.

1 Introduction

The modelling and simulation of traffic flow is a challenging task in particular in order to design infrastructure. Indeed, there are some examples in which the construction of a new infrastructure did not improve the traffic. For example, in Stuttgart, Germany, after investments into the road network in 1969, the traffic situation did not improve until a section of newly build road was closed for traffic again (see [22]). This is known as the Braess' paradox. In the past years, a lot of work has been done concerning the modelling and simulation of traffic flows problems.

Traffic flow can be modelled at different scales depending on the level of details one wants to observe: the microscopic scale (describes the dynamics of each of the vehicles), the macroscopic scale (describes the dynamics of the density of vehicles) and the mesoscopic scale (describes the dynamics of the density of vehicles but the car-to-car interactions are not lost).

Microscopic models are considered more justifiable because the behaviour of every single vehicle can be described with high precision whereas macroscopic models are based on assumptions which are less verifiable. Another way to justify macroscopic models is to derive them from microscopic models by rescaling arguments.

The problem of deriving macroscopic models from microscopic ones has already been studied for models of the type following the leader (i.e. the velocity or the acceleration of each vehicle depends only on the distance to the vehicle in front of it). We refer for example to [3, 8, 16, 17, 23] where the authors rescaled the empirical measure and obtained a scalar conservation law (LWR model). In particular, passing from microscopic to macroscopic model for second-order models was instead investigated in [3, 15], where the Aw-Rascle model is derived as the limit of a second order follow-the-leader model.

[1]Normandie Univ, INSA de Rouen, LMI (EA 3226 - FR CNRS 3335), 76000 Rouen, France, 685 Avenue de l'Université, 76801 St Etienne du Rouvray cedex. France

In this paper we establish a connection between a car-following model and a fluid-dynamic model. This result is a generalization of the results of [13] to a second order microscopic model. We consider a second order microscopic model of *follow-the-leader* type with a local perturbation. In such model, the whole traffic flow is determined by the dynamics of the very first vehicle (the *leader*). We will establish a connection between this second order discrete model and a macroscopic model equivalent to a LWR model. The idea is to rescale the microscopic model, which describes the dynamics of each vehicle individually, in order to get a macroscopic model which describes the dynamics of density of vehicles.

The model we study here is similar to the one considered in [12], but in our work, as in [13], we assume that there is a local perturbation (located at the origin for example) that slows down the vehicles and we want to understand how this local perturbation influences the macroscopic dynamics. Due to this perturbation, it is natural to get an Hamilton-Jacobi equation with a junction condition at the origin and an effective flux limiter. Further, our result is stronger than the one in [13] because our microscopic model is a second order model which is more realistic than the first order model considered in the last paper. From a mathematical point of view the fact of considering a second order model presents many technical difficulties. First, we need to consider a system of two non-local PDEs instead of a single equation [11, 12]. Moreover, the two functions that we consider have to satisfy certain properties that derive from the physical characteristics of the microscopic model and those properties need to be proven for the system of non-local PDEs which is more complicated in the case of a second order model than in the case of a first order model.

Paper organization. The paper is organized as follows. In Section 2, we present the microscopic model for which we will present an homogenization result. In Section 3, we inject the system of ODEs into a system of PDEs and we present our main results. Section 3.3 is dedicated to the definition of the non-local operators which appear in the PDEs given in Section 3. In Section 4, we introduce the notion of viscosity solutions for the considered problems and give stability, existence and uniqueness results. In Sections 5 and 6 we present the correctors necessaries for the proof of convergence which is located in Section 7. Section 8 contains the proof of existence of correctors for the junction, where we use the idea developed in [1, 14] and in the lectures of Lions at the "College de France" [25], which consists in constructing correctors on truncated domains. In Section 9 we show the link between the system of ODEs and the system of PDEs which proof is in Appendix B. Finally in Appendix A we analyse the properties of the microscopic model.

2 A first main result

In this paper, we are interested in a second order microscopic model that can simulate the presence of a local perturbation. In order to do that, we considered a modified version of the model introduced by Bando *et al* in [4]. More precisely, we consider a "follow-the-leader" model of the following form

$$\ddot{U}_j(t) = a\left(V(U_{j+1}(t) - U_j(t)) \cdot \phi\left(U_j(t)\right) - \dot{U}_j(t)\right), \tag{2.1}$$

where U_j denotes the position of the j-th vehicle, $\dot{U}_j$ its velocity and $\ddot{U}_j$ its acceleration. The function ϕ simulates the presence of a local perturbation located at the origin and we denote by r its radius of influence. In this model, a and V represent respectively the drivers sensibility and the optimal velocity function. We make the following assumptions on V, ϕ and on the coefficient a.

Assumption (A)

- (A1) $V : \mathbb{R} \to \mathbb{R}^+$ is Lipschitz continuous, non-negative.
- (A2) V is non-decreasing on $\mathbb{R}$.
- (A3) There exists $h_0 \in (0, +\infty)$ such that for all $h \leq h_0$, $V(h) = 0$.

- (A4) There exists $h_{max} \in (h_0, +\infty)$ such that for all $h \geq h_{max}$, $V(h) = V(h_{max}) =: V_{max}$.
- (A5) The function $p \mapsto pV(-1/p)$ is strictly convex on $[-1/h_0, 0)$.
- (A6) The function $\phi : \mathbb{R} \to (0, 1]$ is Lipschitz continuous and $\phi(x) = 1$ for $|x| \geq r$. We denote by $\phi_0 = \min_{x \in [-r,r]} \phi(x) > 0$.
- (A7)(Monotonicity). $a \geq 4 \|V'\|_\infty \|\phi\|_\infty + 4 \|\phi'\|_\infty \|V\|_\infty$.

Remark 2.1 (Remark on (A6))**.** *In the case $\phi = 0$ on an open interval (therefore $\phi_0 = 0$) all the vehicles left of the perturbation would come to a full stop. This case lacks any interest and therefore we can assume that $\phi_0 > 0$.*

Remark 2.2 (Remark on (A7))**.** *Assumption (A7) yields that for all $(b, x) \in \mathbb{R}^2$, the function*

$$f : z \mapsto \frac{a}{2} z - 2V(b + z)\,\phi(x - z)$$

is non-decreasing. This result is particularly important later in the paper because it implies that the systems we consider later in this work are monotone in the sense of Ishii and Koike [21], which will imply the uniqueness of the solution we consider.

As we said in the introduction, in order to obtain an homogenization result for (3.1), we will inject the system of ODEs into a system of PDEs. To do so, we proceed as in [10, 13] by introducing the rescaled "cumulative distribution function", which is the primitive of the rescaled empirical measure, defined by,

$$\rho^\varepsilon(t, y) = -\varepsilon \left(\sum_{i \geq 0} H(y - \varepsilon U_i(t/\varepsilon)) + \sum_{i < 0} (-1 + H(y - \varepsilon U_i(t/\varepsilon))) \right) \tag{2.2}$$

with

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases} \tag{2.3}$$

The macroscopic model

We define $\overline{H} : \mathbb{R} \to \mathbb{R}$, by

$$\overline{H}(p) = \begin{cases} -p - k_0 & \text{for } p < -k_0, \\ -V\left(\dfrac{-1}{p}\right) \cdot |p| & \text{for } -k_0 \leq p < 0, \\ p & \text{for } p \geq 0. \end{cases} \tag{2.4}$$

Note that such $\overline{H}$ is continuous, coercive and because of (A5), there exists a unique point $p_0 \in [-k_0, 0]$ such that

$$\begin{cases} \overline{H} & \text{is decreasing on } (-\infty, p_0) \\ \overline{H} & \text{is increasing on } (p_0, +\infty), \end{cases} \tag{2.5}$$

and we denote by

$$H_0 := \overline{H}(p_0) = \min_{p \in \mathbb{R}} \overline{H}(p) < 0. \tag{2.6}$$

We want to show that the rescaled "cumulative distribution function" converges to the solution of the following macroscopic model.

$$\begin{cases} u^0_t + \overline{H}(u^0_x) = 0 & \text{for} \quad (t,x) \in (0,+\infty) \times (-\infty, 0) \\ u^0_t + \overline{H}(u^0_x) = 0 & \text{for} \quad (t,x) \in (0,+\infty) \times (0, +\infty) \\ u^0_t + F_{\overline{A}}(u^0_x(t,0^-), u^0_x(t,0^+)) = 0 & \text{for} \quad (t,x) \in (0,+\infty) \times \{0\} \\ u^0(0,x) = u_0(x) & \text{for} \quad x \in \mathbb{R}. \end{cases} \tag{2.7}$$

where $\overline{A}$ has to be determined and $F_{\overline{A}}$ is defined by

$$F_{\overline{A}}(p_-, p_+) = \max\left(\overline{A}, \overline{H}^+(p_-), \overline{H}^-(p_+)\right), \tag{2.8}$$

with

$$\overline{H}^-(p) = \begin{cases} \overline{H}(p) & \text{if } p \le p_0 \\ \overline{H}(p_0) & \text{if } p \ge p_0 \end{cases} \quad \text{and} \quad \overline{H}^+(p) = \begin{cases} \overline{H}(p_0) & \text{if } p \le p_0 \\ \overline{H}(p) & \text{if } p \ge p_0. \end{cases} \tag{2.9}$$

The initial condition u_0 is a function that satisfies

$$-k_0 \le (u_0)_x \le 0 \quad \text{and for all } \varepsilon > 0 \quad \rho^\varepsilon(0,x) = \left\lfloor \frac{u_0(x)}{\varepsilon} \right\rfloor. \tag{2.10}$$

According to [19], for all $\overline{A} \in \mathbb{R}$, there exists a unique solution u^0 of (2.7).

Remark 2.3. *We notice that in the case of traffic flow, (2.7) is equivalent (deriving in space) to a LWR model (see [24, 26]) with a flux limiting condition at the origin. In fact, the fundamental diagram of the model is $pV(1/p)$ and u^0_x corresponds to the density of vehicles.*

Passage from a microscopic to a macroscopic model

The main result of this paper is the following convergence result.

Theorem 2.4 (Passage from a microscopic to a macroscopic model)**.** *Assume (A). There exists a unique $\overline{A} \in [H_0, 0]$ such that the function ρ^ε defined by* (2.2) *converges locally uniformly towards the unique solution of* (2.7)*.*

3 Main results

3.1 Injecting the system of ODEs into a system of PDEs

In the rest of the paper, we will work with an equivalent formulation of (2.1). We borrow the idea from [9, 11, 12] and consider for all $j \in \mathbb{Z}$,

$$\Xi_j(t) = U_j(t) + \frac{1}{\alpha}\dot{U}_j(t) \quad \text{with} \quad \alpha = \frac{a}{2}.$$

Using this new function, we obtain the following system of ODEs equivalent to (2.1) for all $j \in \mathbb{Z}$, for all $t \in (0, +\infty)$,

$$\begin{cases} \dot{U}_j(t) = \alpha\left(\Xi_j(t) - U_j(t)\right) \\ \dot{\Xi}_j(t) = \alpha\left(U_j(t) - \Xi_j(t)\right) + 2V\left(U_{j+1}(t) - U_j(t)\right) \cdot \phi\left(U_j(t)\right). \end{cases} \tag{3.1}$$

In Appendix A, we give some properties of system 3.1, such as maximal velocities of the vehicles and minimal and maximal distance between two consecutive vehicles.

We now introduce the "cumulative distribution function" for $(\Xi_j)_j$, defined by

$$\sigma^\varepsilon(t,y) = -\varepsilon\left(\sum_{i\geq 0} H\left(y-\varepsilon\Xi_i\left(t/\varepsilon\right)\right) + \sum_{i<0}\left(-1+H\left(y-\varepsilon\Xi_i\left(t/\varepsilon\right)\right)\right)\right). \tag{3.2}$$

Under assumption (A), $(\rho^\varepsilon, \sigma^\varepsilon)$ is a discontinuous viscosity solution (see Theorem 3.3) of the following non-local equation, for all $(t,x) \in (0,+\infty)\times\mathbb{R}$,

$$\begin{cases} u_t^\varepsilon + M^\varepsilon\left(\frac{u^\varepsilon}{\varepsilon}(t,x), \left[\frac{\xi^\varepsilon}{\varepsilon}(t,\cdot)\right]\right)(x)\cdot|u_x^\varepsilon| = 0 \\ \xi_t^\varepsilon + L^\varepsilon\left(\frac{x}{\varepsilon}, \frac{\xi^\varepsilon}{\varepsilon}(t,x), \left[\frac{u^\varepsilon}{\varepsilon}(t,\cdot)\right]\right)(x)\cdot|\xi_x^\varepsilon| = 0. \end{cases} \tag{3.3}$$

The definition of M^ε and L^ε is postponed to the next section. We submit equation (3.3) to the following initial condition. For all $x \in \mathbb{R}$,

$$\begin{cases} u^\varepsilon(0,x) = u_0(x) \\ \xi^\varepsilon(0,x) = \xi_0^\varepsilon(x). \end{cases} \tag{3.4}$$

We also assume that the initial condition satisfies the following assumption.

(A0) (Gradient bound) Let $k_0 = 1/h_0$. The functions u_0 and ξ_0^ε are Lipschitz continuous functions, such that

$$-k_0 \leq (u_0)_x \leq 0 \tag{3.5}$$

$$-k_0 \leq (\xi_0^\varepsilon)_x \leq 0, \tag{3.6}$$

and

$$0 \leq \xi_0^\varepsilon(x) - u_0(x) \leq \varepsilon. \tag{3.7}$$

Remark 3.1. *The initial conditions u_0 and ξ_0^ε are "regular" functions such that for all $\varepsilon > 0$ we have*

$$\rho^\varepsilon(0,x) = \left\lfloor \frac{u_0(x)}{\varepsilon} \right\rfloor \quad \text{and} \quad \sigma^\varepsilon(0,x) = \left\lfloor \frac{\xi_0^\varepsilon(x)}{\varepsilon} \right\rfloor. \tag{3.8}$$

For $\varepsilon = 1$, the conditions on the gradients translate the fact that at the initial time there is at least h_0 meters between two consecutive vehicles. In the rest of the paper we are interested in the behaviour of ρ^ε and σ^ε as ε goes to 0. This in fact translates to studying the behaviour of the traffic as the number of vehicles per unit length goes to infinity. For $\varepsilon = 1$ condition (3.7) translate the fact that at initial time the velocity of the vehicles must be bounded so the ordering of the vehicles is kept.

The fact that ξ_0^ε depends on ε comes from the rescaling. In fact, given that σ^ε is the "cumulative distribution function" of $(\Xi_j)_j$ which are defined using the velocity of the vehicles, an ε appears multiplying the velocity when rescaling (see [12, Remark 1.2]). Therefore, ξ_0^ε tends to u_0 as ε goes to zero. Finally, to simplify the notations, we denote by $\xi_0 = \xi_0^\varepsilon$ for $\varepsilon = 1$.

3.2 Convergence result

Theorem 2.4 is a consequence of the following theorems. The proof of Theorem 3.2 is postponed until Section 7 and the proof of Theorem 3.3 is postponed until Section 9.

Theorem 3.2 (Junction condition by homogenization)**.** *Assume (A) and (A0). For $\varepsilon > 0$, let $(u^\varepsilon, \xi^\varepsilon)$ be the solution of (3.3)-(3.4). Then there exists $\bar{A} \in [H_0, 0]$ such that u^ε and ξ^ε converge locally uniformly to the unique viscosity solution u^0 of (2.7).*

Theorem 3.3 (Junction condition by homogenization: application to traffic flow)**.** *Assume (A) and that at the initial time* $(U_i(0), \Xi_i(0))_i$ *satisfies*

$$0 \leq \Xi_i(0) - U_i(0) \leq \frac{V_{max}}{\alpha}, \quad U_{i+1}(0) - \Xi_i(0) \geq h_0, \quad \text{and} \quad U_{i+1}(0) - U_i(0) \leq h_{max}.$$

We define two function u_0 *and* ξ_0^ε *satisfying (A0) such that for all* $\varepsilon > 0$,

$$\rho^\varepsilon(0, x) = \varepsilon \left\lfloor \frac{u_0(x)}{\varepsilon} \right\rfloor \quad \text{and} \quad \sigma^\varepsilon(0, x) = \varepsilon \left\lfloor \frac{\xi_0^\varepsilon(x)}{\varepsilon} \right\rfloor,$$

then there exists a unique $\overline{A} \in [H_0, 0]$ *such that the functions* ρ^ε *and* σ^ε *defined by* (2.2) *and* (3.2) *converge locally uniformly towards the unique solution* u^0 *of* (2.7)*.*

The following theorem ensures that when we use (2.7) we only evaluate the function $\overline{H}$ in the interval $[-k_0, 0]$. The proof of Theorem 3.4 is postponed until Section 7.

Theorem 3.4 (Gradient bound)**.** *Assume (A0)-(A). Let* u^0 *be the unique solution of* (2.7)*, then we have for all* $(t, x) \in [0, T] \times \mathbb{R}$,

$$-k_0 \leq u_x^0 \leq 0,$$

with k_0 *defined in (A0).*

3.3 Definition of the non-local operators

In this section, we clarify equation (3.3). We will give the definition of M and L, and then the definition of M^ε and L^ε. To do this, we first introduce the following functions.

$$E(z) = \begin{cases} -\alpha & \text{if } z \geq 0 \\ 0 & \text{if } z < 0, \end{cases} \quad F(z) = \begin{cases} 1 & \text{if } z < 0 \\ 0 & \text{if } z \geq 0, \end{cases} \quad I(z) = \begin{cases} 1 & \text{if } z \geq -1 \\ 0 & \text{if } z < -1, \end{cases}$$

$$\tilde{E}(z) = \begin{cases} -\alpha & \text{if } z > 0 \\ 0 & \text{if } z \leq 0, \end{cases} \quad \tilde{F}(z) = \begin{cases} 1 & \text{if } z \leq 0 \\ 0 & \text{if } z > 0, \end{cases} \quad \text{and} \quad \tilde{I}(z) = \begin{cases} 1 & \text{if } z > -1 \\ 0 & \text{if } z \leq -1. \end{cases}$$

For $x, p \in \mathbb{R}$, we then define the following non-local operators

$$\begin{aligned} M_p\left(U(x), [\Sigma]\right)(x) &= \int_0^D E(\Sigma(x+z) - U(x) + pz)dz, \\ K_p\left(\Sigma(x), [U]\right)(x) &= \int_0^D F(U(x-z) - \Sigma(x) - pz)dz, \\ N_p\left(\Sigma(x), [U]\right)(x) &= \int_0^D I(U(x+z) - \Sigma(x) + pz)dz, \end{aligned}$$

with $D = h_{max} + 3V_{max}/(2\alpha) + 2r/\phi_0$ (see Appendixes A and B for more details on where the constant D comes from). We can now define L^p. For $x, y \in \mathbb{R}$,

$$\begin{aligned} L_p\left(y, \Sigma(x), [U]\right)(x) =& \alpha K_p\left(\Sigma(x), [U]\right)(x) \\ &- 2V\Big(N_p\left(\Sigma(x), [U]\right)(x) + K_p\left(\Sigma(x), [U]\right)(x)\Big) \cdot \phi\left(y - K_p\left(\Sigma(x), [U]\right)(x)\right). \end{aligned} \tag{3.9}$$

In the same way, we define $\tilde{M}_p, \tilde{K}_p$ and $\tilde{N}_p$ by replacing E, F and I respectively by $\tilde{E}, \tilde{F}$ and $\tilde{I}$. Similary,

$$\begin{aligned} \tilde{L}_p\left(y, \Sigma(x), [U]\right)(x) =& \alpha \tilde{K}_p\left(\Sigma(x), [U]\right)(x) \\ &- 2V\Big(\tilde{N}_p\left(\Sigma(x), [U]\right)(x) + \tilde{K}_p\left(\Sigma(x), [U]\right)(x)\Big) \cdot \phi\left(y - \tilde{K}_p\left(\Sigma(x), [U]\right)(x)\right). \end{aligned} \tag{3.10}$$

For $p = 0$, we define

$$M(U(x),[\Sigma])(x) := M_0(U(x),[\Sigma]) = \int_0^D E(\Sigma(x+z) - U(x))dz, \tag{3.11}$$

$$K\left(\Sigma(x),[U]\right)(x) := K_0\left(\Sigma(x),[U]\right)(x) = \int_0^D F(U(x-z) - \Sigma(x))dz, \tag{3.12}$$

$$N\left(\Sigma(x),[U]\right)(x) := N_0\left(\Sigma(x),[U]\right)(x) = \int_0^D I(U(x+z) - \Sigma(x))dz, \tag{3.13}$$

and

$$\begin{aligned} L\left(y,\Sigma(x),[U]\right)(x) =& \alpha K\left(\Sigma(x),[U]\right)(x) \\ &- 2V\Big(N\left(\Sigma(x),[U]\right)(x) + K\left(\Sigma(x),[U]\right)(x)\Big)\cdot \phi\left(y - K\left(\Sigma(x),[U]\right)(x)\right). \end{aligned} \tag{3.14}$$

Remark 3.5 (Remarks on the non-local operators)**.** *First let us notice that the domain of integration in the non-local operators is bounded by a constant $D := h_{max} + 3V_{max}/(2\alpha) + 2r/\phi_0$, this comes from the fact that the velocities of the vehicles as well as the distance between two consecutive vehicles from model 3.1 are bounded (see Appendix A). In particular, there exists a constant $M_0 > 0$ (independent of p), such that we have the following bounds on the non-local operators,*

$$\begin{aligned} -M_0 \le -\alpha D \le M_p(U(x),[\Sigma])(x) \le 0, \\ M_0 \ge D \ge K_p(\Sigma(x),[U])(x) \ge 0, \\ M_0 \ge D \ge N_p(\Sigma(x),[U])(x) \ge 0, \\ M_0 \ge \alpha D \ge L_p(y,\Sigma(x),[U])(x) \ge -2V_{max} \ge -M_0, \end{aligned}$$

with $M_0 = \max(2V_{max}, \alpha D, D)$.

Finally, we would like to point out that given the fact that the function V is non-decreasing (assumption (A2)) and that the function $F \ge 0$ and therefore $K(\Sigma,[U])(x) \ge 0$, we have

$$L(y,\Sigma(x),[U])(x) \ge -2V\Big(N\left(\Sigma(x),[U]\right)(x)\Big). \tag{3.15}$$

Finally, we introduce for $\varepsilon > 0$,

$$M^\varepsilon\left(U(x),[\Sigma]\right)(x) = \int_0^D E(\Sigma(x+\varepsilon z) - U(x))dz, \tag{3.16}$$

$$K^\varepsilon\left(\Sigma(x),[U]\right)(x) = \int_0^D F(U(x-\varepsilon z) - \Sigma(x))dz, \tag{3.17}$$

$$N^\varepsilon\left(\Sigma(x),[U]\right)(x) = \int_0^D I(U(x+\varepsilon z) - \Sigma(x))dz, \tag{3.18}$$

$$\tag{3.19}$$

and

$$\begin{aligned} L^\varepsilon\left(y,\Sigma(x),[U]\right)(x) =& \alpha K^\varepsilon\left(\Sigma(x),[U]\right)(x) \\ &- 2V\Big(N^\varepsilon\left((\Sigma(x),[U]\right)(x) + K^\varepsilon\left(\Sigma(x),[U]\right)(x)\Big)\cdot \phi\left(y - K^\varepsilon\left(\Sigma(x),[U]\right)(x)\right). \end{aligned} \tag{3.20}$$

The bounds provided by Remark 3.5 remain valid for the non-local operators depending on $\varepsilon > 0$.

Remark 3.6 (Lagrangian formulation)**.** *Another way to treat this problem is to consider a Lagrangian formulation, like in [12], considering the functions,*

$$u(t,y) = U_{\lfloor y \rfloor}(t) \quad \textit{and} \quad \xi(t,y) = \Sigma_{\lfloor y \rfloor}(t).$$

The couple (u,ξ) satisfies for all $(t,y)\in[0,T]\times\mathbb{R}$

$$\begin{cases} u_t(t,y)=\alpha\left(\xi(t,y)-u(t,y)\right)\\ \xi_t(t,y)=\alpha\left(u(t,y)-\xi(t,y)\right)+2V\left(u(t,y+1)-u(t,y)\right)\cdot\phi(u(t,y))\\ u(0,y)=u_0(y)\\ \xi(0,y)=\xi_0^\varepsilon(y).\end{cases}$$

We note that the system we obtain is much more simple. Nevertheless, the difficulty with this formulation is that the function ϕ is evaluated at $u(t,y)$ and not at a physical point of the road. At the macroscopic scale, we then expect to get a junction condition located at $u=0$. The notion of junction in this case is not well defined and this is why we use the formulation (3.3)(where the perturbation function is evaluated at a point of the road). This will allow us to use the results of Imbert and Monneau [19] concerning quasi-convex Hamiltonians with a junction condition.

4 Viscosity Solutions

This section is devoted to the definition and useful results for viscosity solutions of the problems considered in this paper. The reader is referred to the user's guide of Crandall, Ishii, Lions [6] and the book of Barles [5] for an introduction to viscosity solutions. In order to give a general definition, we will give the definition of viscosity solutions for the following equation, with $p\in\mathbb{R}$, and for all $(t,x)\in(0,+\infty)\times\mathbb{R}$,

$$\begin{cases} u_t+\psi(x)\cdot M_p\left(u(t,x),[\xi(t,\cdot)]\right)(x)\cdot|p+u_x|+(1-\psi(x))\cdot\overline{H}\left(u_x\right)=0\\ \xi_t+\psi(x)\cdot L_p\left(x,\xi(t,x),[u(t,\cdot)]\right)(x)\cdot|p+\xi_x|+(1-\psi(x))\cdot\overline{H}\left(\xi_x\right)=0\\ u(0,x)=u_0(x)\\ \xi(0,x)=\xi_0(x),\end{cases}\tag{4.1}$$

with $\psi:\mathbb{R}\to[0,1]$ a Lipschitz continuous function. We also use the following notations for the upper and lower semi-continuous envelopes of a locally bounded function u:

$$u^*(t,x)=\limsup_{s\to t,y\to x}u(s,y)\quad\text{and}\quad u_*(t,x)=\liminf_{s\to t,y\to x}u(s,y).$$

4.1 Definitions

Definition 4.1 (Viscosity solutions for (4.1))**.** *Let $T>0$. Let $u:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ and $\xi:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ be upper semi-continuous (resp. lower semi-continuous) functions. We say that (u,ξ) is a viscosity sub-solution (resp. super-solution) of* (4.1) *on $[0,T]\times\mathbb{R}$ if $u(0,x)\le u_0(x)$ and $\xi(0,x)\le\xi_0(x)$ (resp. $u(0,x)\ge u_0(x)$ and $\xi(0,x)\ge\xi_0(x)$) and for all $(t,x)\in(0,T)\times\mathbb{R}$, and for any test function $\varphi\in C^1((0,T)\times\mathbb{R})$ such that $u-\varphi$ attains a local maximum (resp. a local minimum) at the point (t,x), we have*

$$\varphi_t+\psi(x)\cdot M_p\left(u(t,x),[\xi(t,\cdot)]\right)(x)\cdot|p+\varphi_x|+(1-\psi(x))\cdot\overline{H}(\varphi_x)\le 0,$$
$$(\textit{resp.}\quad \varphi_t+\psi(x)\cdot\tilde{M}_p\left(u(t,x),[\xi(t,\cdot)]\right)(x)\cdot|p+\varphi_x|+(1-\psi(x))\cdot\overline{H}(\varphi_x)\ge 0),$$

and for all $(t,x)\in(0,T)\times\mathbb{R}$ and any test function $\varphi\in C^1((0,T)\times\mathbb{R})$ such that $\xi-\varphi$ attains a local maximum (resp. a local minimum) at the point (t,x), we have

$$\varphi_t+\psi(x)\cdot L_p\left(x,\xi(t,x),[u(t,\cdot)]\right)(x)\cdot|p+\varphi_x|+(1-\psi(x))\cdot\overline{H}(\varphi_x)\le 0,$$

$$(\textit{resp.}\quad \varphi_t+\psi(x)\cdot\tilde{L}_p\left(x,\xi(t,x),[u(t,\cdot)]\right)(x)\cdot|p+\varphi_x|+(1-\psi(x))\cdot\overline{H}(\varphi_x)\ge 0).$$

We say that (u,ξ) is a viscosity solution of (4.1) if (u^*,ξ^*) and (u_*,ξ_*) are respectively a sub-solution and a super-solution of (4.1).

Proposition 4.2 (Stability result for (4.1))**.** *Let* (u_n, ξ_n) *be a sequence of uniformly bounded upper semi-continuous functions (resp. lower semi-continuous) and let* $(p_n)_n$ *be such that* $p_n \to p$. *We assume that* (u_n, ξ_n) *is a sub-solution (resp. a super-solution) of* (4.1) *with* $p = p_n$. *Let* $(\overline{u}, \overline{\xi}) = (\limsup^* u_n, \limsup^* \xi_n)$ *(resp.* $(\underline{u}, \underline{\xi}) = (\liminf_* u_n, \liminf_* \xi_n)$*). Then* $(\overline{u}, \overline{\xi})$ *(resp.* $(\underline{u}, \underline{\xi})$*) is a sub-solution (resp. a super-solution) of* (4.1)*.*

Proof. The proof is classical and we refer to [10]. The only point to note is that both Hamiltonians in (4.1) are monotone with respect to the non-local operators (this is a consequence of assumption (A7) for the non-local operator K_p).

□

4.2 Viscosity solutions for (2.7)

The theory of viscosity solutions for Hamilton-Jacobi equations on networks was recently treated in several papers. We give here some results for viscosity solutions of (2.7) that will be used in the rest of paper and we refer to [19] for the general theory and for the proofs.

Definition 4.3 (Class of test function for (2.7))**.** *We denote by* $J_\infty := (0, +\infty) \times \mathbb{R}$*,* $J_\infty^+ := (0, +\infty) \times (0, +\infty)$ *and* $J_\infty^- := (0, +\infty) \times (-\infty, 0)$ *, we define a class of test function on* J_∞ *by*

$$C^1(J_\infty) = \{\varphi \in C(J_\infty), \text{the restriction of } \varphi \text{ to } J_\infty^+ \text{ and to } J_\infty^- \text{ are } C^1\}.$$

Definition 4.4 (Viscosity solution for (2.7))**.** *An upper semi-continuous (resp. lower semi-continuous) function* $u : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ *is a viscosity sub-solution (resp. super-solution) of (2.7) if* $u(0, x) \le u_0(x)$ *(resp.* $u(0, x) \ge u_0(x)$*) and for all* $(t, x) \in J_\infty$ *and for all* $\varphi \in C^1(J_\infty)$ *such that*

$$u \le \varphi \quad (\text{resp. } u \ge \varphi) \text{ in a neighbourhood of } (t, x) \in J_\infty \text{ and } u(t, x) = \varphi(t, x), \text{ we have}$$

$$\begin{aligned} \varphi_t(t, x) + \overline{H}(\varphi_x(t, x)) &\le 0 \quad (\text{resp. } \ge 0) \qquad \text{if } x \neq 0 \\ \varphi_t(t, x) + F_A(\varphi_x(t, 0^-), \varphi_x(t, 0^+)) &\le 0 \quad (\text{resp. } \ge 0) \qquad \text{if } x = 0. \end{aligned}$$

We say that a function u *is a viscosity solution of (2.7) if* u^* *and* u_* *are respectively a sub-solution and a super-solution of (2.7). We refer to this solution as A-flux-limited solution.*

Now we recall an equivalent definition (Theorem 2.5 in [19]) for sub and super solution at the junction. We will also consider the following problem,

$$u_t + \overline{H}(u_x) = 0 \quad \text{for } t \in (0, T) \text{ and } x \in \mathbb{R}\backslash\{0\}. \tag{4.2}$$

Theorem 4.5 (Equivalent definition for sub/super-solutions)**.** *Let* $\overline{H}$ *given by (2.4) and consider* $A \in [H_0, +\infty)$ *with* H_0 *defined in (2.6). Given arbitrary solutions* $p_\pm^A \in \mathbb{R}$ *of*

$$\overline{H}\left(p_+^A\right) = \overline{H}^+\left(p_+^A\right) = A = \overline{H}^-\left(p_-^A\right) = \overline{H}\left(p_-^A\right), \tag{4.3}$$

let us fix any time independent test function $\phi^0(x)$ *satisfying*

$$\phi_x^0(0^\pm) = p_\pm^A.$$

Given a function $u : (0, T) \times \mathbb{R} \to \mathbb{R}$*, the following properties hold true.*

1. *If* u *is an upper semi-continuous sub-solution of (4.2) satisfying*

$$u(t, 0) = \limsup_{(t,y)\to(t,0), y\in J_i^*} u(s, y), \tag{4.4}$$

   then u *is a* H_0*-flux limited sub-solution.*

2. *Given $A > H_0$ and $t_0 \in (0,T)$, if u is an upper semi-continous sub-solution of (4.2) satisfying* (4.4) *and if for any test function φ touching u from above at $(t_0, 0)$ with*

$$\varphi(t,x) = \psi(t) + \phi^0(x), \tag{4.5}$$

for some $\psi \in C^1(0,+\infty)$, we have

$$\varphi_t + F_A\left(\varphi_x\left(t_0, 0^-\right), \varphi_x\left(t_0, 0^+\right)\right) \leq 0 \quad \text{at } (t_0, 0),$$

then u is a A-flux limited sub-solution at $(t_0, 0)$.

3. *Given $t_0 \in (0,T)$, if u is a lower semi-continuous super-solution of (4.2) and if for any test function φ satisfying (4.5) touching u from above at $(t_0, 0)$ we have*

$$\varphi_t + F_A\left(\varphi_x\left(t_0, 0^-\right), \varphi_x\left(t_0, 0^+\right)\right) \geq 0 \quad \text{at } (t_0, 0),$$

then u is a A-flux limited super-solution at $(t_0, 0)$.

4.3 Existence and uniqueness of viscosity solution for (4.1) with $p = 0$

We recall that for $p = 0$, our equation is

$$\begin{cases} u_t + M(u(t,x), [\xi(t,\cdot)])(x) \cdot |u_x| = 0 & \text{for} \quad (t,x) \in (0,+\infty) \times \mathbb{R}, \\ \xi_t + L\left(x, \xi(t,x), [u(t,\cdot)]\right)(x) \cdot |\xi_x| = 0 & \text{for} \quad (t,x) \in (0,+\infty) \times \mathbb{R}, \\ u(0,x) = u_0(x) & \text{for} \quad x \in \mathbb{R}, \\ \xi(0,x) = \xi_0(x) & \text{for} \quad x \in \mathbb{R}. \end{cases} \tag{4.6}$$

Lemma 4.6 (Existence of barriers for (4.6))**.** *Assume (A) and (A0). There exists a constant $K_1 > 0$ such that*

$$(u^-, \xi^-) = (u_0 - K_1 t, \xi_0 - K_1 t) \quad \text{and} \quad (u^+, \xi^+) = (u_0 + K_1 t, \xi_0 + K_1 t) \tag{4.7}$$

are respectively sub-solution and super-solution of (4.6).

Proof. We define $K_1 = M_0 k_0$. Let us prove that (u^+, ξ^+) is a super-solution of (4.6). In fact, we have that

$$u_t^+ + \tilde{M}(u^+(t,x), [\xi^+(t,\cdot)])(x)|u_x^+| \geq K_1 - M_0 k_0 = 0,$$

where we have used Remark 3.5 for the second inequality. Similarly, using that $\tilde{K} \geq 0$ and $K_1 \geq 2||V||_\infty k_0$, we have that

$$\xi_t^+ + \tilde{L}(x, \xi^+(t,x), [u^+(t,\cdot)])(x)|\xi_t^+| \geq 0.$$

The proof that (u^-, ξ^-) is a sub-solution is similar and we skip it.

□

Proposition 4.7 (Comparaison principle)**.** *Let $T > 0$. Assume (A)-(A0). Let (u, ξ) and (v, ζ) be respectively a sub-solution and a super-solution of (4.6). We also assume that there exists a constant $C > 0$ such that for all $(t,x) \in [0,T] \times \mathbb{R}$, we have*

$$u_0(x) - Ct \leq u(t,x) \leq u_0(x) + Ct, \qquad \xi_0(x) - Ct \leq \xi(t,x) \leq \xi_0(x) + Ct \tag{4.8}$$

and

$$-u_0(x) - Ct \leq -v(t,x) \leq -u_0(x) + Ct, \qquad -\xi_0(x) - Ct \leq -\zeta(t,x) \leq -\xi_0(x) + Ct. \tag{4.9}$$

If

$$u(0,x) \leq v(0,x) \quad \text{and} \quad \xi(0,x) \leq \zeta(0,x) \quad \text{for all } x \in \mathbb{R},$$

then

$$u(t,x) \leq v(t,x) \quad \text{and} \quad \xi(t,x) \leq \zeta(t,x) \quad \text{for all } x \in \mathbb{R}, t \in [0,T].$$

Proof. Let us introduce

$$\overline{M} = \sup_{t\in[0,T],x\in\mathbb{R}} \max\left(u(t,x) - v(t,x), \xi(t,x) - \zeta(t,x)\right).$$

We want to prove that $\overline{M} \leq 0$. We argue by contradiction by assuming that $M > 0$.

Step 1: test functions. We introduce the following test functions

$$\varphi(t,x,y) = u(t,x) - v(t,y) - \frac{\eta}{T-t} - e^{At}\left(\frac{(x-y)^2}{2\varepsilon} + \gamma\frac{x^2}{2}\right)$$

and

$$\psi(t,x,y) = \xi(t,x) - \zeta(t,y) - \frac{\eta}{T-t} - e^{At}\left(\frac{(x-y)^2}{2\varepsilon} + \gamma\frac{x^2}{2}\right),$$

with η, γ small parameters, and A a constant to be chosen later. We denote by $\Phi(t,x,y) = \max\left(\varphi(t,x,y), \psi(t,x,y)\right)$. Using (4.8) and (4.9) we have that

$$\begin{aligned}\varphi(t,x,y) &\leq u_0(x) - u_0(y) + 2CT - \frac{\eta}{T-t} - e^{At}\left(\frac{(x-y)^2}{2\varepsilon} + \gamma\frac{x^2}{2}\right)\\ &\leq 2CT + k_0|x-y| - \frac{\eta}{T-t} - e^{At}\left(\frac{(x-y)^2}{2\varepsilon} + \gamma\frac{x^2}{2}\right).\end{aligned}$$

We have a similar result for ψ which yields that

$$\lim_{|x|,|y|\to+\infty} \Phi = -\infty.$$

Using the fact that our test functions are upper semi continuous, we can see that Φ reaches a maximum at some finite point that we denote by $(\bar{t},\bar{x},\bar{y}) \in [0,T) \times \mathbb{R} \times \mathbb{R}$. Classicaly we have for η and γ small enough,

$$\begin{cases} M_{\eta,\varepsilon,\gamma} = \Phi(\bar{t},\bar{x},\bar{y}) \geq \dfrac{\overline{M}}{2} > 0, \\ |\bar{x}-\bar{y}| \to 0 \quad \text{as} \quad \varepsilon \to 0, \\ \gamma|\bar{x}| \to 0 \quad \text{as} \quad \gamma \to 0. \end{cases}$$

Step 2: $\bar{t} > 0$ for ε small enough. By contradiction, let us assume that Φ reaches its maximum for $\bar{t} = 0$. Let us for instance assume that $\Phi(\bar{t},\bar{x},\bar{y}) = \varphi(\bar{t},\bar{x},\bar{y})$. In this case, we have

$$0 < \frac{\overline{M}}{2} \leq u(0,\bar{x}) - v(0,\bar{y}) - \frac{\eta}{T-t} \leq k_0|\bar{x}-\bar{y}| - \frac{\eta}{T-t}.$$

Therefore, $\dfrac{\eta}{T} < k_0|\bar{x}-\bar{y}|$ and for ε small enough we get a contradiction. In the same way, we get a contradiction if we assume that $\phi(\bar{t},\bar{x},\bar{y}) = \psi(\bar{t},\bar{x},\bar{y})$.

Step 3: utilisation of the equation in the case $\Phi(\bar{t},\bar{x},\bar{y}) = \varphi(\bar{t},\bar{x},\bar{y})$. By duplication of the time variable and passing to the limit we have that there exist two real numbers $a, b \in \mathbb{R}$ such that

$$a - b = \frac{\eta}{(T-\bar{t})^2} + Ae^{A\bar{t}}\left(\frac{(\bar{x}-\bar{y})^2}{2\varepsilon} + \gamma\frac{\bar{x}^2}{2}\right) \tag{4.10}$$

$$a + M(u(\bar{t},\bar{x}), [\xi(\bar{t},\cdot)])(\bar{x})|e^{A\bar{t}}\left(p_\varepsilon + \gamma\bar{x}\right)| \leq 0 \tag{4.11}$$

$$b+\tilde{M}(v(\bar{t},\bar{y}),[\zeta(\bar{t},\cdot)])(\bar{y})|e^{A\bar{t}}p_\varepsilon|\geq 0 \tag{4.12}$$

with $p_\varepsilon=\dfrac{\bar{x}-\bar{y}}{\varepsilon}$. Combining (4.10), (4.11) and (4.12), we obtain

$$\frac{\eta}{(T-\bar{t})^2}+Ae^{A\bar{t}}\left(\frac{(\bar{x}-\bar{y})^2}{2\varepsilon}+\gamma\frac{\bar{x}^2}{2}\right)\leq |e^{A\bar{t}}p_\varepsilon|\Big(\tilde{M}(v(\bar{t},\bar{y}),[\zeta(\bar{t},\cdot)])(\bar{y})-M(u(\bar{t},\bar{x}),[\xi(\bar{t},\cdot)])(\bar{x})\Big)+o_\gamma, \tag{4.13}$$

where we have used the fact that $M(u(\bar{t},\bar{x}),[\xi(\bar{t},\cdot)])(\bar{x})$ is finite according to Remark 3.5.

We distinguish two cases.

Case 1: there exists a subsequence γ_n such that

$$\frac{|\bar{x}-\bar{y}|}{\varepsilon}\to 0\quad \text{as } n\to+\infty.$$

In this case, taking γ going to zero in (4.13) yields a contradiction.

Case 2: there exists a constant $C_\varepsilon>0$ such that for any γ small enough we have,

$$\frac{|\bar{x}-\bar{y}|}{\varepsilon}\geq C_\varepsilon.$$

Changing variables in (4.13) we can write

$$\begin{aligned}
\frac{\eta}{(T-\bar{t})^2}+Ae^{A\bar{t}}\frac{(\bar{x}-\bar{y})^2}{2\varepsilon} &\leq |e^{A\bar{t}}p_\varepsilon|\int_{\bar{y}}^{D+\bar{y}}\tilde{E}(\zeta(\bar{t},z)-v(\bar{t},\bar{y}))dz\\
&\quad -|e^{A\bar{t}}p_\varepsilon|\int_{\bar{x}}^{D+\bar{x}}E(\xi(\bar{t},z)-u(\bar{t},\bar{x}))dz+o_\gamma(1)\\
&\leq |e^{A\bar{t}}p_\varepsilon|\int_{\bar{y}}^{D+\bar{y}}\tilde{E}(\zeta(\bar{t},z)-v(\bar{t},\bar{y}))-E(\xi(\bar{t},z)-u(\bar{t},\bar{x}))dz\\
&\quad +|e^{A\bar{t}}p_\varepsilon|\left|\int_{\bar{y}}^{\bar{x}}E(\xi(\bar{t},z)-u(\bar{t},\bar{x}))dz\right|\\
&\quad +|e^{A\bar{t}}p_\varepsilon|\left|\int_{D+\bar{x}}^{D+\bar{y}}E(\xi(\bar{t},z)-u(\bar{t},\bar{x}))dz\right|+o_\gamma(1).
\end{aligned} \tag{4.14}$$

We define

$$\mathcal{A}=\left\{z\in\mathbb{R}:\tilde{E}\left(\zeta(\bar{t},z)-v(\bar{t},\bar{y})\right)\leq E\left(\xi(\bar{t},z)-u(\bar{t},\bar{x})\right)\right\}.$$

The inequality $\varphi(\bar{t},\bar{x},\bar{y})\geq\psi(\bar{t},z,z)$ yields

$$\zeta(\bar{t},z)-v(\bar{t},\bar{y})\geq\xi(\bar{t},z)-u(\bar{t},\bar{x})+e^{A\bar{t}}\left(\frac{(\bar{x}-\bar{y})^2}{2\varepsilon}+\gamma\frac{\bar{x}^2}{2}-\gamma\frac{z^2}{2}\right).$$

This implies that

$$\mathcal{A}^c\subset\{|z|\geq R_{\varepsilon,\gamma}\}\quad \text{with } R_{\varepsilon,\gamma}^2=\frac{2}{\gamma}\left(\frac{(\bar{x}-\bar{y})^2}{2\varepsilon}+\gamma\frac{\bar{x}^2}{2}\right).$$

Moreover $\tilde{R}_{\varepsilon,\gamma}=R_{\varepsilon,\gamma}-|\bar{y}|\to+\infty$ as $\gamma\to 0$ (see Da Lio *et al.* in [7, Lemma 2.5]). This implies that

$$\begin{aligned}
\int_{\bar{y}}^{D+\bar{y}}\tilde{E}(\zeta(\bar{t},z)-v(\bar{t},\bar{y}))dz &=\int_{[\bar{y},D+\bar{y})\cap\mathcal{A}}\tilde{E}(\zeta(\bar{t},z)-v(\bar{t},\bar{y}))dz\\
&\quad +\int_{[\bar{y},D+\bar{y})\cap\mathcal{A}^c}\tilde{E}(\zeta(\bar{t},z)-v(\bar{t},\bar{y}))dz.
\end{aligned}$$

However, from Remark 3.5, we have that for γ small enough

$$\begin{aligned}0 \leq \int_{[\bar{y}, D+\bar{y}) \cap \mathcal{A}^c} -\tilde{E}(\zeta(\bar{t}, z) - v(\bar{t}, \bar{y}])dz &= \int_{[\bar{y}, D+\bar{y}] \cap \{|z| \geq R_{\varepsilon,\gamma}\}} -\tilde{E}(\zeta(\bar{t}, z) - v(\bar{t}, \bar{y}))dz \\ &= \int_{[0,D] \cap \{|z+\bar{y}| \geq R_{\varepsilon,\gamma}\}} -\tilde{E}(\zeta(\bar{t}, z+\bar{y}) - v(\bar{t}, \bar{y}))dz \\ &\leq \int_{[0,D] \cap \{|z| \geq \tilde{R}_{\varepsilon,\gamma}\}} -\tilde{E}(\zeta(\bar{t}, z+\bar{y}) - v(\bar{t}, \bar{y}))dz \\ &= 0.\end{aligned}$$

We deduce that for γ small enough,

$$\int_{\bar{y}}^{D+\bar{y}} \tilde{E}(\zeta(\bar{t}, z) - v(\bar{t}, \bar{y}))dz = \int_{\bar{y}}^{D+\bar{y}} E(\xi(\bar{t}, z) - u(\bar{t}, \bar{x}))dz.$$

Then for γ small enough (4.14) implies

$$\begin{aligned}\frac{\eta}{(T-\bar{t})^2} + Ae^{A\bar{t}} \frac{(\bar{x}-\bar{y})^2}{2\varepsilon} \leq& \left|e^{A\bar{t}} p_\varepsilon\right| \left| \int_{\bar{y}}^{\bar{x}} E(\xi(\bar{t}, z) - u(\bar{t}, \bar{x}))dz + \int_{D+\bar{x}}^{D+\bar{y}} E(\xi(\bar{t}, z) - u(\bar{t}, \bar{x}))dz \right| + o_\gamma \\ \leq& 2\alpha e^{A\bar{t}} \frac{(\bar{x}-\bar{y})^2}{\varepsilon} + o_\gamma.\end{aligned}$$

Choosing $A = 4\alpha$, we get a contradiction.

Step 4: utilisation of equation in the case $\Phi(\bar{t}, \bar{x}, \bar{y}) = \psi(\bar{t}, \bar{x}, \bar{y})$. By duplication of the time variable and passing to the limit, we have that there exist two real numbers $a, b \in \mathbb{R}$ such that

$$a - b = \frac{\eta}{(T-\bar{t})^2} + Ae^{A\bar{t}} \left(\frac{(\bar{x}-\bar{y})^2}{2\varepsilon} + \gamma \frac{\bar{x}^2}{2} \right) \tag{4.15}$$

$$a + L(\bar{x}, \xi(\bar{t}, \bar{x}), [u(\bar{t}, \cdot)])(\bar{x}) \cdot |e^{A\bar{t}}(p_\varepsilon + \gamma \bar{x})| \leq 0 \tag{4.16}$$

$$b + \tilde{L}(\bar{y}, \xi(\bar{t}, \bar{y}), [u(\bar{t}, \cdot)])(\bar{y}) \cdot |e^{A\bar{t}} p_\varepsilon| \geq 0 \tag{4.17}$$

with $p_\varepsilon = \frac{\bar{x}-\bar{y}}{\varepsilon}$. Combining (4.15), (4.16) and (4.17), we obtain that

$$\frac{\eta}{(T-\bar{t})^2} + Ae^{A\bar{t}} \frac{(\bar{x}-\bar{y})^2}{2\varepsilon} \leq |e^{A\bar{t}} p_\varepsilon| \left(\tilde{L}(\bar{y}, \xi(\bar{t}, \bar{y}), [u(\bar{t}, \cdot)])(\bar{y}) - L(\bar{x}, \xi(\bar{t}, \bar{x}), [u(\bar{t}, \cdot)])(\bar{x}) \right) + o_\gamma. \tag{4.18}$$

We recall that we defined L and $\tilde{L}$ using K and V (see (3.9) and (3.10)). Therefore, we can see that the right part of inequality (4.18) is finite (using Remark 3.5). We distinguish two cases.

Case 1: there exists a subsequence γ_n such that

$$\frac{|\bar{x}-\bar{y}|}{\varepsilon} \to 0 \quad \text{as } n \to +\infty.$$

In this case, taking γ to zero in (4.18) yields a contradiction.

Case 2: there exists a constant $C_\varepsilon > 0$, such that for any γ small enough we have

$$\frac{|\bar{x}-\bar{y}|}{\varepsilon} \geq C_\varepsilon.$$

To simplify, we introduce

$$\begin{array}{ll} L = L(\bar{x}, \xi(\bar{t},\bar{x}), u[(\bar{t},\cdot)])(\bar{x}) & \tilde{L} = \tilde{L}(\bar{y}, \zeta(\bar{t},\bar{y}), [v(\bar{t},\cdot)])(\bar{y}), \\ K = K(\xi(\bar{t},\bar{x}), [u(\bar{t},\cdot)])(\bar{x}) & \tilde{K} = \tilde{K}(\zeta(\bar{t},\bar{y}), [v(\bar{t},\cdot)])(\bar{y}), \\ N = N(\xi(\bar{t},\bar{x}), [u(\bar{t},\cdot)])(\bar{x}) & \tilde{N} = \tilde{N}(\zeta(\bar{t},\bar{y}), [v(\bar{t},\cdot)])(\bar{y}). \end{array}$$

As above, we can prove

$$\tilde{K} - K \leq |\bar{x}-\bar{y}| \quad \text{and} \quad N - \tilde{N} \leq |\bar{x}-\bar{y}|.$$

We have that

$$\begin{aligned} \tilde{L} - L &= \alpha \tilde{K} - 2V\left(\tilde{N} + \tilde{K}\right)\phi\left(\bar{y} - \tilde{K}\right) - L \\ &\leq \alpha\left(K + |\bar{x}-\bar{y}|\right) - 2V\left(\tilde{N} + K + |\bar{x}-\bar{y}|\right)\phi\left(\bar{y} - K - |\bar{x}-\bar{y}|\right) - L \\ &\leq \alpha\left(K + |\bar{x}-\bar{y}|\right) - 2V\left(N + K\right)\phi\left(\bar{y} - K - |\bar{x}-\bar{y}|\right) - L \\ &\leq \alpha|\bar{x}-\bar{y}| + 2V\left(N + K\right)\left(\phi\left(\bar{x} - K\right) - \phi\left(\bar{y} - K - |\bar{x}-\bar{y}|\right)\right) \\ &\leq \alpha|\bar{x}-\bar{y}| + 2\left|\left|V\right|\right|_\infty \left|\left|\phi'\right|\right|_\infty |\bar{x}-\bar{y}|, \end{aligned} \tag{4.19}$$

where we have used for the first inequality the monotonicity (see Remark 2.2). The monotonicity of V yields the second inequality. The third and the final inequalities come from the definition of L and the fact that ϕ and V are Lipschitz functions. Finally, combining (4.19) with (4.18), we obtain

$$\frac{\eta}{(T-\bar{t})^2} + Ae^{A\bar{t}}\frac{(\bar{x}-\bar{y})^2}{2\varepsilon} \leq e^{A\bar{t}}\frac{(\bar{x}-\bar{y})^2}{\varepsilon}\left(\alpha + 2\left|\left|V\right|\right|_\infty \left|\left|\phi'\right|\right|_\infty\right) + o_\gamma(1). \tag{4.20}$$

Taking $A = 2\left(\alpha + 2\left|\left|V\right|\right|_\infty \left|\left|\phi'\right|\right|_\infty\right)$, we get a contradiction in (4.20). The proof of Proposition 4.7 is now complete.

□

We now give a comparison principle on bounded sets, to do this, we define for a given point $(t_0, x_0) \in (0, T) \times \mathbb{R}$ and for $\overline{r}, \overline{R} > 0$, the set

$$\mathcal{Q}_{\overline{r},\overline{R}}(t_0, x_0) = (t_0 - \overline{r}, t_0 + \overline{r}) \times (x_0 - \overline{R}, x_0 + \overline{R}).$$

Proposition 4.8 (Comparison principle on bounded sets for (4.6))**.** *Assume (A). Let (u, ξ) be a sub-solution of (4.6) and let (v, ζ) be a super-solution of (4.6) on the open set $\mathcal{Q}_{\overline{r},\overline{R}} \subset (0, T) \times \mathbb{R}$. Also assume that*

$$u \leq v \quad \textit{and} \quad \xi \leq \zeta \quad \textit{outside } \mathcal{Q}_{\overline{r},\overline{R}},$$

then

$$u \leq v \quad \textit{and} \quad \xi \leq \zeta \quad \textit{on } \mathcal{Q}_{\overline{r},\overline{R}}.$$

Applying Perron's method (see [20, Proof of Theorem 6], [2] or [18] to see how to apply Perron's method for problems with non-local terms), joint to the comparison principle, we obtain the following result.

Theorem 4.9 (Existence and uniqueness of viscosity solutions for (4.6))**.** *Assume (A0) and (A). Then, there exists a unique solution (u, ξ) of (4.6). Moreover, the functions u and ξ are continuous and there exists a constant $K_1 > 0$ such that*

$$u_0(x) - K_1 t \leq u(t, x) \leq u_0(x) + K_1 t \quad \textit{and} \quad \xi_0(x) - K_1 t \leq \xi(t, x) \leq \xi_0(x) + K_1 t. \tag{4.21}$$

4.4 Control of the oscillations for (4.6)

We now present a theorem that provides a control on the oscillations in space of the solution of (4.6). This is a very important theorem, first because it will allow us to prove Theorem 3.4 and also because it presents some of the arguments we use later to build the correctors at the junction.

Theorem 4.10 (Control of the space oscillations)**.** *Let $T > 0$. Assume (A0)-(A) and let (u, ξ) be the solution of (4.6) provided by Theorem 4.9. Then for all $x, y \in \mathbb{R}, x \geq y$ and for all $t \in [0, T]$, we have*

$$-k_0(x-y) - 1 \leq u(t,x) - u(t,y) \leq 0 \tag{4.22}$$

and

$$-k_0(x-y) - 1 \leq \xi(t,x) - \xi(t,y) \leq 0, \tag{4.23}$$

with k_0 defined in (A0).

Proof. We use the following notation,

$$\Omega = \{(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R} \quad \text{s.t. } x \geq y\}.$$

Proof of the upper bound. We introduce

$$\overline{M} = \sup_{(t,x,y)\in\Omega} \max\left(u(t,x) - u(t,y), \xi(t,x) - \xi(t,y)\right).$$

We want to prove that $\overline{M} \leq 0$. We argue by contradiction and assume that $\overline{M} > 0$.

Step 1: the test functions. For $\eta, \gamma > 0$ small parameters, we define

$$\varphi(t,x,y) = u(t,x) - u(t,y) - \frac{\eta}{T-t} - \gamma x^2 - \gamma y^2$$

and

$$\psi(t,x,y) = \xi(t,x) - \xi(t,y) - \frac{\eta}{T-t} - \gamma x^2 - \gamma y^2.$$

We denote by $\Phi(t,x,y) = \max\left(\varphi(t,x,y), \psi(t,x,y)\right)$. For $x \geq y$, using (4.21) and (A0) we have

$$\varphi(t,x,y) \leq u_0(x) - u_0(y) + 2K_1T - \frac{\eta}{T-t} - \gamma x^2 - \gamma y^2 \leq 2K_1T - \gamma x^2 - \gamma y^2$$

$$\psi(t,x,y) \leq \xi_0(x) - \xi_0(y) + 2K_1T - \frac{\eta}{T-t} - \gamma x^2 - \gamma y^2 \leq 2K_1T - \gamma x^2 - \gamma y^2.$$

Therefore, we deduce

$$\lim_{|x|,|y|\to+\infty} \Phi(t,x,y) = -\infty.$$

Since φ, ψ are upper semi continuous, Φ reaches a maximum on Ω at a point that we denote by $(\bar{t}, \bar{x}, \bar{y})$. Classically we have for η and γ small enough

$$\begin{cases} 0 < \dfrac{\overline{M}}{2} \leq \Phi(\bar{t}, \bar{x}, \bar{y}), \\ \gamma|\bar{x}|, \gamma|\bar{y}| \to 0 \quad \text{as } \gamma \to 0. \end{cases}$$

Step 2: $\bar t > 0$ and $\bar x > \bar y$. By contradiction, assume first that $\bar t = 0$. For instance, we assume that $\Phi(\bar t, \bar x, \bar y) = \varphi(\bar t, \bar x, \bar y)$. In this case, we have that

$$\frac{\eta}{T} \leq u_0(\bar x) - u_0(\bar y) \leq 0,$$

where we have used the fact that u_0 is non increasing, and we get a contradiction. In the same way, we get a contradiction if $\Phi(\bar t, \bar x, \bar y) = \psi(\bar t, \bar x, \bar y)$. The fact that $\bar x > \bar y$ is obtained directly using that $\Phi(\bar t, \bar x, \bar y) > 0$.

Step 3: utilisation of the equation in the case $\Phi(\bar t, \bar x, \bar y) = \varphi(\bar t, \bar x, \bar y)$. By duplication of the time variable and passing to the limit we get that

$$\frac{\eta}{T^2} \leq \frac{\eta}{(T-\bar t)^2} \leq -M(u(\bar t, \bar x), [\xi(\bar t, \cdot)])(\bar x) \cdot |2\gamma \bar x|, \tag{4.24}$$

where we have used the fact that $\tilde M(u(\bar t, \bar y), [\xi(\bar t, \cdot)])(\bar y) \leq 0$. Using Remark 3.5, we have that $-M(u(\bar t, \bar x), [\xi(\bar t, \cdot)])(\bar x)$ is bounded. Taking γ to zero, we get a contradiction in (4.24).

Step 4: utilisation of equation in the case $\Phi(\bar t, \bar x, \bar y) = \psi(\bar t, \bar x, \bar y)$. By duplication of the time variable and passing to the limit we get that

$$\begin{aligned}\frac{\eta}{(T-\bar t)^2} &\leq \tilde L(\bar y, \xi(\bar t, \bar y), [u(\bar t, \cdot)])(\bar y)|2\gamma \bar y| - L(\bar x, \xi(\bar t, \bar x), [u(\bar t, \cdot)])(\bar x)|2\gamma \bar x| \\ &\leq 2M_0\left(|\gamma \bar x| + |\gamma \bar y|\right)\end{aligned}$$

where we have used the bounds on L and $\tilde L$ (see Remark 3.5). Taking γ to zero, we get a contradiction.

Proof of the lower bound. In order to prove our result, we will use the following lemma which proof is postponed.

Lemma 4.11. *For all $(t, x) \in [0, T] \times \mathbb{R}$, we have*

$$0 \leq \xi(t, x) - u(t, x) \leq 1. \tag{4.25}$$

Now we would like to prove that for all $\varepsilon > 0$,

$$\overline{M} = \sup_{(t,x,y)\in\Omega} \{\xi(t, y) - u(t, x) - (k_0 + \varepsilon)(x - y) - 1\} \leq 0. \tag{4.26}$$

In fact, if (4.26) is true, then taking ε to 0 and using (4.25) we directly obtain the lower inequalities in (4.22) and (4.23). We argue by contradiction and assume that $\overline{M} > 0$.

Step 1: the test function. For $\eta, \gamma > 0$ small parameters, we define

$$\varphi(t, x, y) = \xi(t, y) - u(t, x) - (k_0 + \varepsilon)(x - y) - 1 - \frac{\eta}{T - t} - \gamma x^2.$$

Using (A0) and (4.21), we obtain that

$$\begin{aligned}\varphi(t, x, y) &\leq \xi_0(y) - u_0(x) + 2K_1 T - (k_0 + \varepsilon)(x - y) - 1 - \frac{\eta}{T - t} - \gamma x^2 \\ &\leq 1 + k_0(x - y) + 2K_1 T - (k_0 + \varepsilon)(x - y) - 1 - \frac{\eta}{T - t} - \gamma x^2 \\ &\leq 2K_1 T - \gamma x^2 - \varepsilon(x - y).\end{aligned}$$

Therefore, we have that for $(t,x,y)\in\Omega$

$$\lim_{|x|,|y|\to+\infty}\varphi(t,x,y)=-\infty.$$

Since φ is upper semi continuous, φ reaches a maximum on Ω at a point that we denote by $(\bar t,\bar x,\bar y)$. Classically we have for η and γ small enough

$$\begin{cases}0<\dfrac{\overline{M}}{2}\le\varphi(\bar t,\bar x,\bar y),\\ \gamma|\bar x|\to 0\quad\text{as }\gamma\to 0.\end{cases}$$

Step 2: $\bar t>0$ and $\bar x>\bar y$. By contradiction, assume first that $\bar t=0$. Using (A0), we get a contradiction writing that

$$\frac{\eta}{T}<\xi_0(\bar y)-u_0(\bar x)-(k_0+\varepsilon)(\bar x-\bar y)-1\le 1+k_0(\bar x-\bar y)-(k_0+\varepsilon)(\bar x-\bar y)-1\le 0.$$

If we assume that $\bar x=\bar y$ then, using the fact that $\varphi(\bar t,\bar x,\bar y)>0$, we get that

$$0<\xi(\bar t,\bar x)-u(\bar t,\bar x)-1-\frac{\eta}{T-\bar t}\le 1-1-\frac{\eta}{T}=-\frac{\eta}{T}.$$

This inequality yields a contradiction.

Step 3: utilisation of the equation. By duplication of the time variable and passing to the limit we get that

$$\begin{aligned}\frac{\eta}{(T-t)^2}&\le\tilde M(u(t,\bar x),[\xi(t,\cdot)])(\bar x)\cdot|2\gamma\bar x+k_0+\varepsilon|-L(\bar y,\xi(\bar t,\bar y),[u(\bar t,\cdot)])(\bar y)\cdot|k_0+\varepsilon|\\&\le -L(\bar y,\xi(\bar t,\bar y),[u(\bar t,\cdot)])(\bar y)\cdot|k_0+\varepsilon|,\end{aligned}$$

where we have used the fact that $\tilde M(u(\bar t,\bar x),[\xi(\bar t,\cdot)])(\bar x)\le 0$. We replace L by its definition (3.14) and using (3.15), we have

$$\frac{\eta}{(T-\bar t)^2}\le 2V\left(N(\xi(\bar t,\bar y),[u(\bar t,\cdot)])(\bar y)\right)|k_0+\varepsilon|. \tag{4.27}$$

Now we want to prove that $N(\xi(\bar t,\bar y),[u(\bar t,\cdot)])(\bar y)\le h_0$. Indeed, if it is true, we will get a contradiction in (4.27) because $V(h)=0\ \forall h\le h_0$. Let then $z>h_0$.

If $\bar y+z\ge\bar x$, then using that $u(\bar t,.)$ is non increasing, we get that

$$u(\bar t,\bar y+z)-\xi(\bar t,\bar y)\le u(\bar t,\bar x)-\xi(\bar t,\bar y)<-k_0(\bar x-\bar y)-1<-1.$$

If $\bar y+z<\bar x$, using the fact that $\varphi(\bar t,\bar x,\bar y+z)\le\varphi(\bar t,\bar x,\bar y)$, we obtain

$$\xi(\bar t,\bar y+z)-\xi(\bar t,\bar y)<-k_0z\le -1.$$

Using Lemma 4.11, we get that $u(\bar t,\bar y+z)-\xi(\bar t,\bar y)<-1$. We deduce that $I(u(\bar t,\bar y+z)-\xi(\bar t,\bar y))=0$ for $z\ge h_0$ and so $N(\xi(\bar t,\bar y),[u(\bar t,.)])(\bar y)\le h_0$.

□

We now turn to the proof of Lemma 4.11.

Proof of Lemma 4.11. The proof is divided into several steps.

Step 1: proof of the lower bound. We introduce

$$\overline{M}=\sup_{(t,x)\in[0,T]\times\mathbb{R}}\left\{u(t,x)-\xi(t,x)\right\}.$$

We want to prove that $\overline{M}\le 0$ and argue by contradiction by assuming that $\overline{M}>0$.

Step 1.1: the test function. Let η, γ be small parameters, and A a constant to be chosen later. We introduce

$$\varphi(t,x,y) = u(t,x) - \xi(t,y) - \frac{\eta}{T-t} - e^{At}\frac{(x-y)^2}{2\varepsilon} - \gamma x^2.$$

Classically, φ reaches a maximum on $[0,T]\times\mathbb{R}\times\mathbb{R}$ at $(\bar{t},\bar{x},\bar{y})$ and we have for η,γ small enough,

$$\begin{cases} 0 < \dfrac{\overline{M}}{2} \le \varphi(\bar{t},\bar{x},\bar{y}), \\ \gamma\bar{x} \to 0 \quad \text{as } \gamma \to 0, \\ |\bar{x}-\bar{y}| \to 0 \quad \text{as } \varepsilon \to 0. \end{cases}$$

Step 1.2: $\bar{t} > 0$ for ε small enough. We assume by contradiction that $\bar{t} = 0$. We have that

$$0 < u_0(\bar{x}) - \xi_0(\bar{y}) - \frac{\eta}{T} \le k_0|\bar{x}-\bar{y}| - \frac{\eta}{T}.$$

Taking ε small enough, we get a contradiction.

Step 1.3: utilisation of equation. By duplication of the time variable and passing to the limit, we get that

$$\begin{aligned} \frac{\eta}{(T-\bar{t})^2} + Ae^{A\bar{t}}\frac{(\bar{x}-\bar{y})^2}{2\varepsilon} &\le \left|e^{A\bar{t}}\frac{\bar{x}-\bar{y}}{\varepsilon}\right| \big(\tilde{L}(\bar{y},\xi(\bar{t},\bar{y}),[u(\bar{t},\cdot)])(\bar{y}) - M(u(\bar{t},\bar{x}),[\xi(\bar{t},\cdot)])(\bar{x})\big) + o_\gamma(1) \\ &\le \left|e^{A\bar{t}}\frac{\bar{x}-\bar{y}}{\varepsilon}\right| \big(\alpha\tilde{K}(\xi(\bar{t},\bar{y}),[u(\bar{t},\cdot)])(\bar{y}) - M(u(\bar{t},\bar{x}),[\xi(\bar{t},\cdot)])(\bar{x})\big) + o_\gamma(1), \end{aligned} \tag{4.28}$$

where we have used the fact that $V \ge 0$. We claim that

$$-M(u(\bar{t},\bar{x}),[\xi(\bar{t},\cdot)])(\bar{x}) \le \alpha|\bar{x}-\bar{y}| \quad \text{and} \quad \tilde{K}(\xi(\bar{t},\bar{y}),[u(\bar{t},\cdot)])(\bar{y}) \le |\bar{x}-\bar{y}|. \tag{4.29}$$

Indeed, for $z > |\bar{x}-\bar{y}|$, using that $\xi(\bar{t},\cdot)$ is non increasing and that $\varphi(\bar{t},\bar{x},\bar{y}) > 0$, we have that

$$\xi(\bar{t},\bar{x}+z) - u(\bar{t},\bar{x}) \le \xi(\bar{t},\bar{y}) - u(\bar{t},\bar{x}) < 0.$$

Therefore, using the definition of E we obtain that (for ε small enough such that $|\bar{x}-\bar{y}| \le D$)

$$-M(u(\bar{t},\bar{x}),[\xi(\bar{t},\cdot)])(\bar{x}) = -\int_0^{|\bar{x}-\bar{y}|} E(\xi(\bar{t},\bar{x}+z) - u(\bar{t},\bar{x}))dz \le \alpha|\bar{x}-\bar{y}|.$$

Similarly, using the fact that $u(\bar{t},\cdot)$ is non increasing, for all $z > |\bar{x}-\bar{y}|$, we have that

$$u(\bar{t},\bar{y}-z) - \xi(\bar{t},\bar{y}) \ge u(\bar{t},\bar{x}) - \xi(\bar{t},\bar{y}) > 0.$$

Therefore,

$$\tilde{K}(\xi(\bar{t},\bar{y}),[u(\bar{t},\cdot)])(\bar{y}) = \int_0^{|\bar{x}-\bar{y}|} \tilde{F}(u(\bar{t},\bar{y}-z) - \xi(\bar{t},\bar{y}))dz \le |\bar{x}-\bar{y}|.$$

This ends the proof of (4.29). Injecting (4.29) into (4.28), we get that

$$\frac{\eta}{(T-\bar{t})^2} + Ae^{A\bar{t}}\frac{(\bar{x}-\bar{y})^2}{2\varepsilon} \le 2\alpha e^{A\bar{t}}\frac{(\bar{x}-\bar{y})^2}{\varepsilon} + o_\gamma(1).$$

Taking $A = 4\alpha$, we get a contradiction for γ small enough.

Step 2: proof of the upper bound. We introduce

$$\overline{M} = \sup_{(t,x)\in[0,T]\times\mathbb{R}} \{\xi(t,x) - u(t,x) - 1\}.$$

We want to prove that $\overline{M} \le 0$. We argue by contradiction and assume that $\overline{M} > 0$.

Let η, γ be small parameters. We consider

$$\varphi(t,x,y) = \xi(t,x) - u(t,y) - 1 - \frac{\eta}{T-t} - \frac{(x-y)^2}{2\varepsilon} - \gamma x^2.$$

Classically, φ reaches a maximum on $[0,T]\times\mathbb{R}\times\mathbb{R}$ at $(\bar{t},\bar{x},\bar{y})$ and we have the following result for η and γ small enough

$$\begin{cases} 0 < \dfrac{\overline{M}}{2} \le \varphi(\bar{t},\bar{x},\bar{y}), \\ |\gamma\bar{x}| \to 0 \quad \text{as } \gamma \to 0, \\ |\bar{x}-\bar{y}| \to 0 \quad \text{as } \varepsilon \to 0. \end{cases} \tag{4.30}$$

As in the previous Step 1.2, we get that $\bar{t} > 0$.

By duplication of the time variable and passing to the limit we then get that

$$\begin{aligned} \frac{\eta}{(T-\bar{t})^2} &\le \left(\tilde{M}(u(\bar{t},\bar{y}),[\xi(\bar{t},\cdot)])(\bar{y}) - L(\bar{x},\xi(\bar{t},\bar{x}),[u(\bar{t},\cdot)])(\bar{x})\right)\left|\frac{\bar{x}-\bar{y}}{\varepsilon}\right| + o_\gamma(1) \\ &\le 2V\left(N(\xi(\bar{t},\bar{x}),[u(\bar{t},\cdot)])(\bar{x})\right)\left|\frac{\bar{x}-\bar{y}}{\varepsilon}\right| + o_\gamma(1), \end{aligned} \tag{4.31}$$

where we have used the fact that $\tilde{M} \le 0$ and (3.15). We want to prove that $N(\xi(\bar{t},\bar{x}),[u(\bar{t},\cdot)])(\bar{x}) \le h_0$. In fact for all $z \ge h_0$, we have that $\bar{x}+z > \bar{y}$ for ε small enough, so using that $\varphi(\bar{t},\bar{x},\bar{y}) > 0$ we get that

$$u(\bar{t},\bar{x}+z) - \xi(\bar{t},\bar{x}) \le u(\bar{t},\bar{y}) - \xi(\bar{t},\bar{x}) < -1.$$

We deduce that $N(\xi(\bar{t},\bar{x}),[u(\bar{t},.)])(\bar{x}) = \int_0^{h_0} I(u(\bar{t},\bar{x}+z) - \xi(\bar{t},\bar{x}))dz \le h_0$. Using that $V(h) = 0$ for $h \le h_0$, we get a contradiction in (4.31) for γ small enough.

□

5 Effective Hamiltonian and effective flux-limiter

In this section we provide a justification for the definition of the effective Hamiltonian $\overline{H}$ provided in (2.4), we use the following proposition.

Proposition 5.1. *(Homogenization left and right of the perturbation) . Assume (A). Then for every $p \in [-k_0, 0]$, there exists a unique $\lambda \in \mathbb{R}$, such that there exists a bounded solution (w,χ) of*

$$\begin{cases} M_p\left(w(x),[\chi]\right)(x)|p+w_x| = \lambda \\ \Big(\alpha K_p\left(\chi(x),[w]\right)(x) - 2V\Big(N_p\left(\chi(x),[w]\right)(x) + K_p\left(\chi(x),[w]\right)(x)\Big)\Big)|p+\chi_x| = \lambda \end{cases} \quad x \in \mathbb{R} \tag{5.1}$$

Moreover, for $p \in [-k_0, 0]$, we have $\lambda = \overline{H}(p) = -V\left(\dfrac{-1}{p}\right)|p|$.

Proof. We claim that $(w,\chi) = \left(0, -\dfrac{p}{\alpha}V\left(\dfrac{-1}{p}\right)\right)$ is a solution of (5.1) for $\lambda = -|p|V\left(\dfrac{-1}{p}\right)$.

-If $p = 0$, the result is obvious.

-If $p \in [-k_0, 0)$, since $\frac{-p}{\alpha} V\left(\frac{-1}{p}\right) + pz \geq 0$ if and only if $z \in [0, V(-1/p)/\alpha]$, then we have

$$M_p(w(x), [\chi])(x) = \int_0^D E\left(-\frac{p}{\alpha} V\left(\frac{-1}{p}\right) + pz\right) dz = -V\left(\frac{-1}{p}\right), \tag{5.2}$$

we recall that $D = h_{max} + 3V_{max}/(2\alpha) + 2r/\phi_0$. Similarly, for all $z > 0$, we have $\frac{p}{\alpha} V\left(\frac{-1}{p}\right) - pz < 0$ if and only if $z < V(-1/p)/\alpha$, then

$$K_p(\chi(x), [w])(x) = \int_0^D F\left(\frac{p}{\alpha} V\left(\frac{-1}{p}\right) - pz\right) dz = \frac{1}{\alpha} V\left(\frac{-1}{p}\right). \tag{5.3}$$

Finally, by definition we have that

$$N_p(\chi(x), [w])(x) = \int_0^D I\left(\frac{p}{\alpha} V\left(\frac{-1}{p}\right) + pz\right) dz.$$

First, notice that thanks to assumption (A7), for all $p \in [-k_0, 0)$, we have $\frac{1}{\alpha} V\left(\frac{-1}{p}\right) + \frac{1}{p} < 0$. Moreover, $\frac{p}{\alpha} V\left(\frac{-1}{p}\right) + pz > -1$ for $z < \frac{-1}{\alpha} V\left(\frac{-1}{p}\right) - \frac{1}{p}$. We distinguish two cases.

Case 1: $\frac{-1}{\alpha} V\left(\frac{-1}{p}\right) - \frac{1}{p} \leq D$. In this case, we have

$$N_p(\chi(x), [w])(x) = \frac{-1}{\alpha} V\left(\frac{-1}{p}\right) - \frac{1}{p}$$

and

$$N_p(\chi(x), [w])(x) + K_p(\chi(x), [w])(x) = -\frac{1}{p}. \tag{5.4}$$

Finally, using (5.2), (5.3), and (5.4), we obtain our desired result.

Case 2: $\frac{-1}{\alpha} V\left(\frac{-1}{p}\right) - \frac{1}{p} > D$. In particular we have $-1/p \geq h_{max}$. Therefore, we have

$$N_p(\chi(x), [w])(x) = D \quad \text{and} \quad K_p(\chi(x), [w])(x) = \frac{V_{max}}{\alpha},$$

this implies that

$$N_p(\chi(x), [w])(x) + K_p(\chi(x), [w])(x) = D + \frac{V_{max}}{\alpha} > h_{max}.$$

Combining this result to (5.3), we obtain

$$\begin{aligned}\Big(\alpha K_p(\chi(x), [w])(x) - 2V\Big(N_p(\chi(x), [w])(x) + K_p(\chi(x), [w])(x)\Big)\Big)|p| &= -V_{max}|p| \\ &= -V\left(\frac{-1}{p}\right) \cdot |p|.\end{aligned} \tag{5.5}$$

Using (5.2) and (5.5), we obtain our desired result. The proof is now complete.

□

6 Correctors for the junction

In order to obtain an homogenization result, we need to find the effective flux-limiter. That is why we consider the following cell problem: find $\lambda \in \mathbb{R}$ such that there exists a solution (w, χ) of the following Hamilton-Jacobi equation, for $x \in \mathbb{R}$,

$$\begin{cases} M\left(w(x), [\chi(\cdot)]\right) \cdot |w_x| = \lambda \\ L\left(x, \chi(x), [w(\cdot)]\right)(x) \cdot |\chi_x| = \lambda. \end{cases} \tag{6.1}$$

In this section we present a result of existence of correctors for the junction, which will be used for the proof of Theorem 3.2. We use the following notation: given $\overline{A} \in \mathbb{R}$, $\overline{A} \geq H_0$, we define two real numbers $\overline{p}_-$ and $\overline{p}_+$ defined by

$$\overline{H}(\overline{p}_+) = \overline{H}^+(\overline{p}_+) = \overline{H}(\overline{p}_-) = \overline{H}^-(\overline{p}_-) = \overline{A}. \tag{6.2}$$

Given the form of $\overline{H}$, there exists only one couple of real numbers satisfying (6.2).

Theorem 6.1 (Existence of global corrector for the junction)**.** *Assume (A).*

i) (General properties) There exists a constant $\overline{A} \in [H_0, 0]$ such that there exists a solution (w, χ) of (6.1) with $\lambda = \overline{A}$ and such that there exists a constant $C > 0$ and a globally Lipschitz continuous function m such that for all $x \in \mathbb{R}$,

$$|w(x) - m(x)| \leq C \quad \textit{and} \quad |\chi(x) - m(x)| \leq C. \tag{6.3}$$

ii) (Bound from below at infinity) If $\overline{A} > H_0$, then there exists a $\gamma_0 > 0$ such that for every $\gamma \in (0, \gamma_0)$, we have for all $x \geq r + V_{max}/\alpha$ and $h \geq 0$,

$$\begin{aligned} w(x+h) - w(x) &\geq (\overline{p}_+ - \gamma)h, \\ \chi(x+h) - \chi(x) &\geq (\overline{p}_+ - \gamma)h \end{aligned} \tag{6.4}$$

and for $x \leq -r - V_{max}/\alpha$ and $h \geq 0$,

$$\begin{aligned} w(x-h) - w(x) &\geq (-\overline{p}_- - \gamma)h, \\ \chi(x-h) - \chi(x) &\geq (-\overline{p}_- - \gamma)h. \end{aligned} \tag{6.5}$$

(iii) (Rescaling) For $\varepsilon > 0$, we set

$$w^\varepsilon(x) = \varepsilon w\left(\frac{x}{\varepsilon}\right) \quad \textit{and} \quad \chi^\varepsilon(x) = \varepsilon \chi\left(\frac{x}{\varepsilon}\right),$$

then (up to a sub-sequence $\varepsilon_n \to 0$) we have that w^ε and χ^ε converge locally uniformly towards a function W which satisfies

$$\begin{cases} |W(x) - W(y)| \leq C|x - y| & \textit{for all } x, y \in \mathbb{R}, \\ \overline{H}(W_x) = \overline{A} & \textit{for all } x \neq 0. \end{cases} \tag{6.6}$$

In particular, we have (with $W(0) = 0$),

$$W(x) = \overline{p}_+ x 1_{\{x>0\}} + \overline{p}_- x 1_{\{x<0\}}. \tag{6.7}$$

The proof of this theorem is postponed until Section 8.

7 Proof of convergence

This section is devoted to the proof of Theorem 3.2 which relies on the existence of correctors provided by Proposition 5.1 and Theorem 6.1. We will use the following lemmas, the first one being a direct consequence of Theorem 4.9.

Lemma 7.1. *(Barriers uniform in ε). Assume (A0) and (A). There exist a constant $K_1 > 0$ such that for all $t \geq 0$ and $x \in \mathbb{R}$, we have*

$$|u^\varepsilon(t,x) - u_0(x)| \leq K_1 t \quad \textit{and} \quad |\xi^\varepsilon(t,x) - \xi_0(x)| \leq K_1 t \tag{7.1}$$

The following lemma is a direct consequence of Theorem 4.10.

Lemma 7.2. *(Uniform gradient bound). Assume (A0) and (A). Then the solution $(u^\varepsilon, \xi^\varepsilon)$ of (3.3) satisfies for all $t \geq 0$, for all $x, y \in \mathbb{R}$, $x \geq y$,*

$$\begin{aligned} -k_0(x-y) - \varepsilon \leq u^\varepsilon(t,x) - u^\varepsilon(t,y) \leq 0, \\ -k_0(x-y) - \varepsilon \leq \xi^\varepsilon(t,x) - \xi^\varepsilon(t,y) \leq 0. \end{aligned} \tag{7.2}$$

Before passing to the proof of Theorem 3.2, let us mention that Theorem 3.4 is a direct consequence of this result joint to Theorem 3.2.

We now turn to the proof of Theorem 3.2.

Proof of Theorem 3.2. We introduce

$$\begin{aligned} \overline{u}(t,x) = \limsup_{\varepsilon \to 0}{}^* u^\varepsilon \quad \overline{\xi}(t,x) = \limsup_{\varepsilon \to 0}{}^* \xi^\varepsilon, \\ \underline{u}(t,x) = \liminf_{\varepsilon \to 0}{}_* u^\varepsilon \quad \underline{\xi}(t,x) = \liminf_{\varepsilon \to 0}{}_* \xi^\varepsilon, \end{aligned}$$

and

$$\overline{v} = \max\left(\overline{u}, \overline{\xi}\right) \quad \underline{v} = \min\left(\underline{u}, \underline{\xi}\right).$$

We want to prove that $\overline{v}$ is a sub-solution of (2.7) and that $\underline{v}$ is a super-solution of (2.7). Indeed, in this case, the comparison principle will imply that $\overline{v} \leq \underline{v}$. But by construction $\underline{v} \leq \overline{v}$, hence $\overline{v} = \underline{v} = u^0$, the unique solution of (2.7). This implies that $\overline{u} = \underline{u} = \overline{\xi} = \underline{\xi} = u^0$ and so u^ε and ξ^ε converge locally uniformly to u^0.

To prove that $\overline{v}$ is a sub-solution of (2.7), we argue by contradiction and assume that there is a point $(\bar{t}, \bar{x}) \in (0, +\infty) \times \mathbb{R}$ and a test function $\varphi \in C^1(J_\infty)$ such that

$$\begin{cases} \overline{v}(\bar{t}, \bar{x}) = \varphi(\bar{t}, \bar{x}), \\ \overline{v} \leq \varphi \quad \text{on} \quad Q_{\bar{r},\bar{r}}(\bar{t}, \bar{x}) & \text{with } \bar{r} > 0, \\ \overline{v} \leq \varphi - 2\eta \quad \text{outside} \quad Q_{\bar{r},\bar{r}}(\bar{t}, \bar{x}) & \text{with } \eta > 0, \\ \varphi_t(\bar{t}, \bar{x}) + \overline{H}\left(\bar{x}, \varphi_x(\bar{t}, \bar{x})\right) = \theta > 0, \end{cases} \tag{7.3}$$

where

$$\overline{H}\left(\bar{x}, \varphi_x(\bar{t}, \bar{x})\right) = \begin{cases} \overline{H}\left(\varphi_x(\bar{t}, \bar{x})\right) & \text{if } \bar{x} \neq 0 \\ F_{\overline{A}}\left(\varphi_x(\bar{t}, 0^-), \varphi_x(\bar{t}, 0^+)\right) & \text{if } \bar{x} = 0. \end{cases}$$

We can assume that for ε small enough (up to changing φ at infinity), we have

$$u^\varepsilon, \xi^\varepsilon \leq \varphi - \eta \quad \text{outside} \quad Q_{\bar{r},\bar{r}}(\bar{t}, \bar{x}). \tag{7.4}$$

Using Lemmas 7.1 and 7.2 we get that the functions $\overline{u}$ and $\overline{\xi}$ satisfy for all $t > 0$,

$$|\overline{u}(t,x) - u_0(x)| \leq K_1 t \quad \text{and} \quad |\overline{\xi}(t,x) - \xi_0(x)| \leq K_1 t \quad \text{for all } x \in \mathbb{R}, \tag{7.5}$$

and

$$-k_0(x-y) \leq \overline{u}(t,x) - \overline{u}(t,y) \leq 0 \quad \text{and} \quad -k_0(x-y) \leq \overline{\xi}(t,x) - \overline{\xi}(t,y) \leq 0 \quad \text{for} \quad x \geq y. \tag{7.6}$$

We distinguish two cases.

Case 1: $\bar{x} \neq 0$. We only consider the case $\bar{x} > 0$, since the other case ($\bar{x} < 0$) is treated in the same way. We define $p = \varphi_x(\bar{t}, \bar{x})$, that according to (7.6), satisfies

$$-k_0 \leq p \leq 0 \tag{7.7}$$

We choose $\bar{r}$ small enough so that $\bar{x} - 2\bar{r} > 0$. We introduce

$$\psi^\varepsilon(t, x) = \varphi(t, x) - \varepsilon \frac{p}{\alpha} V\left(\frac{-1}{p}\right).$$

We have the following lemma.

Lemma 7.3. *$(\varphi, \psi^\varepsilon)$ satisfies, in the viscosity sense, the inequality*

$$\begin{cases} \varphi_t + \tilde{M}^\varepsilon\left(\frac{\varphi}{\varepsilon}(t,x), \left[\frac{\psi^\varepsilon}{\varepsilon}(t,\cdot)\right]\right)(x) \cdot |\varphi_x| \geq \frac{\theta}{2} \\ \psi^\varepsilon_t + \tilde{L}^\varepsilon\left(\frac{x}{\varepsilon}, \frac{\psi^\varepsilon}{\varepsilon}(t,x), \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x) \cdot |\psi^\varepsilon_x| \geq \frac{\theta}{2} \end{cases} \quad \text{on } Q_{\bar{r},\bar{r}}(\bar{t},\bar{x}). \tag{7.8}$$

Proof of Lemma 7.3. For all $(t,x) \in Q_{\bar{r},\bar{r}}(\bar{t},\bar{x})$, we have for $\bar{r}$ small enough

$$\begin{aligned} \varphi_t(t,x) + \tilde{M}^\varepsilon\left(\frac{\varphi}{\varepsilon}(t,x), \left[\frac{\psi^\varepsilon}{\varepsilon}(t,\cdot)\right]\right)(x) \cdot |\varphi_x(t,x)| =& \varphi_t(\bar{t},\bar{x}) + o_{\bar{r}}(1) \\ &+ \tilde{M}^\varepsilon\left(\frac{\varphi}{\varepsilon}(t,x), \left[\frac{\psi^\varepsilon}{\varepsilon}(t,\cdot)\right]\right)(x) \cdot |p| \\ =& \theta + o_{\bar{r}}(1) \\ &+ \tilde{M}^\varepsilon\left(\frac{\varphi}{\varepsilon}(t,x), \left[\frac{\psi^\varepsilon}{\varepsilon}(t,\cdot)\right]\right)(x) \cdot |p| - \overline{H}(p) \\ =:& \Delta \end{aligned}$$

where we have used for the first equality the regularity of the test function φ and the fact that the non-local operator $\tilde{M}^\varepsilon$ is bounded (see Remark 3.5) and (7.3) for the second equality.

If $p = 0$, we obtain directly our result. We then assume that $p \in [-k_0, 0)$. For all $D \geq z \geq 0$, and for ε and $\bar{r}$ small enough we have that

$$\frac{\psi^\varepsilon(t, x + \varepsilon.z) - \varphi(t,x)}{\varepsilon} \leq pz - \frac{p}{\alpha} V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_\varepsilon(1),$$

where we have used the fact that $\varphi \in C^1$. Now using the fact that $\tilde{E}$ is non increasing, we have

$$\tilde{E}\left(\frac{\psi^\varepsilon(t, x + \varepsilon.z) - \varphi(t,x)}{\varepsilon}\right) \geq \tilde{E}\left(pz - \frac{p}{\alpha} V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_\varepsilon(1)\right). \tag{7.9}$$

Moreover, we have that

$$pz - \frac{p}{\alpha} V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_\varepsilon(1) \geq 0 \quad \text{iff} \quad z \leq \frac{1}{\alpha} V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_\varepsilon(1).$$

We deduce that

$$\begin{aligned} \tilde{M}^\varepsilon\left(\frac{\varphi}{\varepsilon}(t,x), \left[\frac{\psi^\varepsilon}{\varepsilon}(t,\cdot)\right]\right)(x) \geq& \int_0^D \tilde{E}\left(pz - \frac{p}{\alpha} V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_\varepsilon(1)\right) dz \\ \geq& -V\left(-\frac{1}{p}\right) + o_{\bar{r}}(1) + o_\varepsilon(1). \end{aligned} \tag{7.10}$$

Using (7.9),(7.10) and the definition of $\overline{H}$, we have for $\bar{r}$ and ε small enough,

$$\Delta \geq \theta + o_{\bar{r}}(1) - V\left(\frac{-1}{p}\right)|p| + o_{\bar{r}}(1) + o_{\varepsilon}(1) + V\left(\frac{-1}{p}\right)|p| = \theta + o_{\bar{r}}(1) + o_{\varepsilon}(1) \geq \frac{\theta}{2}.$$

We now prove the second inequality in (7.8). Let us notice that for ε small enough, using the fact that the non-local operator $\tilde{K}^{\varepsilon}$ is bounded (see Remark 3.5) and the definition of ϕ, we have that

$$\phi\left(\frac{x}{\varepsilon} - \tilde{K}^{\varepsilon}\left(\frac{\psi^{\varepsilon}}{\varepsilon}(t,x), \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)\right) = 1 \quad \text{for all } (t,x) \in Q_{\bar{r},\bar{r}}(\bar{t},\bar{x}).$$

For all $(t,x) \in Q_{\bar{r},\bar{r}}(\bar{t},\bar{x})$, we have for $\bar{r}$ small enough

$$\begin{aligned}
\psi^{\varepsilon}_t(t,x) + \tilde{L}^{\varepsilon}\left(\frac{x}{\varepsilon}, \frac{\psi^{\varepsilon}}{\varepsilon}(t,x), \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)|\psi^{\varepsilon}_x(t,x)| =& \varphi_t(t,x) \\
&+ \tilde{L}^{\varepsilon}\left(\frac{x}{\varepsilon}, \frac{\psi^{\varepsilon}}{\varepsilon}(t,x), \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)|\varphi_x(t,x)| \\
=& \theta + o_{\bar{r}}(1) \\
&+ \tilde{L}^{\varepsilon}\left(\frac{x}{\varepsilon}, \frac{\psi^{\varepsilon}}{\varepsilon}(t,x), \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)|p| - \overline{H}(p) \\
=:& \Delta'
\end{aligned}$$

If $p = 0$, we obtain directly our result. We then assume that $p \in [-k_0, 0)$. For all $D \geq z \geq 0$, and for ε and $\bar{r}$ small enough we have that

$$\frac{\varphi(t, x - \varepsilon z) - \psi^{\varepsilon}(t,x)}{\varepsilon} \leq -pz + \frac{p}{\alpha}V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_{\varepsilon}(1).$$

Now, using the fact that $\tilde{F}$ is non increasing, we have that

$$\int_0^D \tilde{F}\left(-pz + \frac{p}{\alpha}V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_{\varepsilon}(1)\right)dz \leq \tilde{K}^{\varepsilon}\left(\frac{\psi^{\varepsilon}}{\varepsilon}(t,x), \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)$$

which yields that

$$\frac{1}{\alpha}V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_{\varepsilon}(1) \leq \tilde{K}^{\varepsilon}\left(\frac{\psi^{\varepsilon}}{\varepsilon}(t,x), \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x). \tag{7.11}$$

We now compute $\tilde{N}^{\varepsilon}\left(\frac{\psi^{\varepsilon}}{\varepsilon}(t,x), \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)$. As above, and using the fact that $\tilde{I}$ is non decreasing, we have

$$\tilde{N}^{\varepsilon}\left(\frac{\psi^{\varepsilon}}{\varepsilon}(t,x), \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x) \leq \int_0^D \tilde{I}\left(pz + \frac{p}{\alpha}V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_{\varepsilon}(1)\right)dz. \tag{7.12}$$

We notice that thanks to assumption (A7), for all $p \in [-k_0, 0)$ we have $\frac{1}{p} + \frac{1}{\alpha}V\left(\frac{-1}{p}\right) < 0$. Using that $pz + \frac{p}{\alpha}V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_{\varepsilon}(1) > -1$ if and only if $z < -\frac{1}{p} - \frac{1}{\alpha}V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_{\varepsilon}(1)$, we have distinguish two cases.

First case: $-\frac{1}{p}-\frac{1}{\alpha}V\left(\frac{-1}{p}\right)+o_{\bar r}(1)+o_\varepsilon(1)\le D$**.** In this case,

$$\begin{aligned}\tilde N^\varepsilon\left(\frac{\psi^\varepsilon}{\varepsilon}(t,x),\left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)\le&\int_0^D\tilde I\left(pz+\frac{p}{\alpha}V\left(\frac{-1}{p}\right)+o_{\bar r}(1)+o_\varepsilon(1)\right)dz\\ \le&-\frac{1}{p}-\frac{1}{\alpha}V\left(\frac{-1}{p}\right)+o_{\bar r}(1)+o_\varepsilon(1).\end{aligned}\tag{7.13}$$

Then,

$$\begin{aligned}\Delta'\ge&\theta+o_{\bar r}(1)+\tilde L^\varepsilon\left(\frac{x}{\varepsilon},\frac{\psi^\varepsilon}{\varepsilon}(t,x),\left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)|p|-\overline{H}(p)\\
\ge&\theta+o_{\bar r}(1)+\alpha\tilde K^\varepsilon\left(\frac{\psi^\varepsilon}{\varepsilon}(t,x),\left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)\\
&-2V\left(\tilde N^\varepsilon\left(\frac{\psi^\varepsilon}{\varepsilon}(t,x),\left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)+\tilde K^\varepsilon\left(\frac{\psi^\varepsilon}{\varepsilon}(t,x),\left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)\right)+V\left(\frac{-1}{p}\right)|p|\\
\ge&\theta+o_{\bar r}(1)+V\left(\frac{-1}{p}\right)+o_{\bar r}(1)+o_\varepsilon(1)\\
&-2V\left(\tilde N^\varepsilon\left(\frac{\psi^\varepsilon}{\varepsilon}(t,x),\left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)+\frac{1}{\alpha}V\left(\frac{-1}{p}\right)+o_{\bar r}(1)+o_\varepsilon(1)\right)+V\left(\frac{-1}{p}\right)|p|\\
\ge&\theta+o_{\bar r}(1)+V\left(\frac{-1}{p}\right)|p|+o_{\bar r}(1)+o_\varepsilon(1)-2V\left(\frac{-1}{p}+o_{\bar r}(1)+o_\varepsilon(1)\right)|p|+V\left(\frac{-1}{p}\right)|p|\\
\ge&\theta+o_{\bar r}(1)+o_\varepsilon(1)\ge\frac{\theta}{2},\end{aligned}$$

where we have used the definition of $\tilde L^\varepsilon$ for the second inequality, (7.11) combined with assumption (A7) (see Remark 2.2) for the third inequality, (7.13) combined with the fact that V is non-decreasing for the fourth inequality and the fact V is a Lipschitz continuous function for the last inequality.

Second case: $-\frac{1}{p}-\frac{1}{\alpha}V\left(\frac{-1}{p}\right)+o_{\bar r}(1)+o_\varepsilon(1)>D$**.** In particular, by definition of D, we have $-1/p\ge h_{max}$ for ε and $\bar r$ small enough. Then using (7.11) and the definition of $\tilde N^\varepsilon$, we obtain

$$\tilde N^\varepsilon\left(\frac{\psi^\varepsilon}{\varepsilon}(t,x),\left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)\le D\quad\text{and}\quad\frac{V_{max}}{\alpha}+o_{\bar r}(1)+o_\varepsilon(1)\le\tilde K^\varepsilon\left(\frac{\psi^\varepsilon}{\varepsilon}(t,x),\left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x).$$

Using assumption (A7) (see Remark 2.2) and the previous inequalities, we get, using the definition of $\tilde L^\varepsilon$, that

$$\begin{aligned}\tilde L^\varepsilon\left(\frac{x}{\varepsilon},\frac{\psi^\varepsilon}{\varepsilon}(t,x),\left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)=&\alpha\tilde K^\varepsilon\left(\frac{\psi^\varepsilon}{\varepsilon}(t,x),\left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)\\
&-2V\left(\tilde N^\varepsilon\left(\frac{\psi^\varepsilon}{\varepsilon}(t,x),\left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)+\tilde K^\varepsilon\left(\frac{\psi^\varepsilon}{\varepsilon}(t,x),\left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)\right)\\
\ge&V_{max}+o_{\bar r}(1)+o_\varepsilon(1)-2V\left(D+\frac{V_{max}}{\alpha}+o_{\bar r}(1)+o_\varepsilon(1)\right)\\
\ge&-V_{max}+o_{\bar r}(1)+o_\varepsilon(1).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\Delta' \geq & \theta + o_{\bar r}(1) - V_{max}|p| + o_{\bar r}(1) + o_\varepsilon(1) + V\left(\frac{-1}{p}\right)|p| \\ \geq & \theta + o_{\bar r}(1) + o_\varepsilon(1) \\ \geq & \frac{\theta}{2},\end{aligned}$$

where we have used assumption (A4) ($V(h) = V_{max}\ \forall h \geq h_{max}$) and that $-1/p \geq h_{max}$. This ends the proof of Lemma 7.3.

□

Getting a contradiction. Using (7.4), we have for ε small enough,

$$u^\varepsilon \leq \varphi - \eta \quad \text{and} \quad \xi^\varepsilon \leq \psi^\varepsilon - \eta \quad \text{outside } Q_{\bar r,\bar r}(\bar t,\bar x).$$

Using the comparison principle on bounded subsets for (3.3), we get

$$u^\varepsilon \leq \varphi - \eta \quad \text{and} \quad \xi^\varepsilon \leq \psi^\varepsilon - \eta \quad \text{on } Q_{\bar r,\bar r}(\bar t,\bar x).$$

Passing to the limit as $\varepsilon \to 0$, we get $\overline{u} \leq \varphi - \eta$ and $\overline{\xi} \leq \varphi - \eta$ on $Q_{\bar r,\bar r}(\bar t,\bar x)$ and this contradicts the fact that $\overline{v}(\bar t,\bar x) = \max\left(\overline{u}(\bar t,\bar x), \overline{\xi}(\bar t,\bar x)\right)) = \varphi(\bar t,\bar x)$.

Case 2: $\bar x = 0$. Using Theorem 4.5, we may assume that the test function has the following form

$$\varphi(t,x) = g(t) + \overline{p}_- x 1_{\{x<0\}} + \overline{p}_+ x 1_{\{x>0\}} \quad \text{on } Q_{\bar r,2\bar r}(\bar t,0), \tag{7.14}$$

where g is a C^1 function defined on $(0,+\infty)$. The last line in condition (7.3) then becomes

$$g'(\bar t) + F_{\overline{A}}(\overline{p}_-,\overline{p}_+) = g'(\bar t) + \overline{A} = \theta.$$

Let us consider (w,ζ) the solution of (6.1) provided by Theorem 6.1. We define

$$\varphi^\varepsilon(t,x) = \begin{cases} g(t) + w^\varepsilon(x) & \text{on } Q_{\bar r,2\bar r}(\bar t,0) \\ \varphi(t,x) & \text{outside } Q_{\bar r,2\bar r}(\bar t,0), \end{cases} \tag{7.15}$$

$$\psi^\varepsilon(t,x) = \begin{cases} g(t) + \zeta^\varepsilon(x) & \text{on } Q_{\bar r,2\bar r}(\bar t,0) \\ \varphi(t,x) & \text{outside } Q_{\bar r,2\bar r}(\bar t,0). \end{cases} \tag{7.16}$$

We have the following lemma,

Lemma 7.4. *$(\varphi^\varepsilon,\psi^\varepsilon)$ satisfies in the viscosity sence, for $\bar r$ and ε small enough on $Q_{\bar r,\bar r}(\bar t,0)$,*

$$\begin{cases} \varphi^\varepsilon_t + \tilde M^\varepsilon\left(\frac{\varphi^\varepsilon}{\varepsilon}(t,x), \left[\frac{\psi^\varepsilon}{\varepsilon}(t,\cdot)\right]\right)(x)\cdot|\varphi^\varepsilon_x| \geq \frac{\theta}{2} \\ \psi^\varepsilon_t + \tilde L^\varepsilon\left(\frac{x}{\varepsilon}, \frac{\psi^\varepsilon}{\varepsilon}(t,x), \left[\frac{\varphi^\varepsilon}{\varepsilon}(t,\cdot)\right]\right)(x)\cdot|\psi^\varepsilon_x| \geq \frac{\theta}{2}. \end{cases} \tag{7.17}$$

Proof of Lemma 7.4. Let h be a test function touching φ^ε from below at $(t_1,x_1) \in Q_{\bar r,\bar r}(\bar t,0)$, so we have

$$w\left(\frac{x_1}{\varepsilon}\right) = \frac{1}{\varepsilon}\left(h(t_1,x_1) - g(t_1)\right)$$

and

$$w(y) \geq \frac{1}{\varepsilon}\left(h(t_1, \varepsilon y) - g(t_1)\right),$$

for y in a neighbourhood of $\frac{x_1}{\varepsilon}$. Since w does not depend on time, we have that

$$h_t(t_1, x_1) = g'(t_1).$$

Using that (w, ζ) is a solution of (6.1), we then deduce that

$$h_t(t_1, x_1) - g'(t_1) + \tilde{M}\left(w\left(\frac{x_1}{\varepsilon}\right), [\zeta]\right)\left(\frac{x_1}{\varepsilon}\right) \cdot |h_x(t_1, x_1)| \geq \overline{A},$$

which implies

$$h_t(t_1, x_1) + \tilde{M}\left(w\left(\frac{x_1}{\varepsilon}\right), [\zeta]\right)\left(\frac{x_1}{\varepsilon}\right) \cdot |h_x(t_1, x_1)| \geq \overline{A} + g'(t_1) \geq \frac{\theta}{2},$$

i.e.

$$h_t(t_1, x_1) + \tilde{M}^\varepsilon\left(\frac{\varphi^\varepsilon}{\varepsilon}(t_1, x_1), \left[\frac{\psi^\varepsilon}{\varepsilon}(t_1, \cdot)\right]\right)(x_1) \cdot |h_x(t_1, x_1)| \geq \frac{\theta}{2}. \tag{7.18}$$

Let f be a test function touching ψ^ε from below at $(t_2, x_2) \in Q_{\bar{r},\bar{r}}(\bar{t}, 0)$. We have

$$\zeta\left(\frac{x_2}{\varepsilon}\right) = \frac{1}{\varepsilon}\left(f(t_2, x_2) - g(t_2)\right)$$

and

$$\zeta(y) \geq \frac{1}{\varepsilon}\left(f(t_2, \varepsilon y) - g(t_2)\right)$$

for y in a neighbourhood of $\frac{x_2}{\varepsilon}$. Since ζ does not depend on time, we have that

$$f_t(t_2, x_2) = g'(t_2).$$

Therefore, using that (w, ζ) is a solution of (6.1), we get

$$f_t(t_2, x_2) - g'(t_2) + \tilde{L}\left(\frac{x_2}{\varepsilon}, \zeta\left(\frac{x_2}{\varepsilon}\right), [w]\right)\left(\frac{x_2}{\varepsilon}\right) \cdot |f_x(t_2, x_2)| \geq \overline{A},$$

which implies

$$f_t(t_2, x_2) + \tilde{L}\left(\frac{x_2}{\varepsilon}, \zeta\left(\frac{x_2}{\varepsilon}\right), [w]\right)\left(\frac{x_2}{\varepsilon}\right) \cdot |f_x(t_2, x_2)| \geq \overline{A} + g_t(t_2) \geq \frac{\theta}{2},$$

i.e.

$$f_t(t_2, x_2) + \tilde{L}^\varepsilon\left(\frac{x_2}{\varepsilon}, \frac{\psi^\varepsilon}{\varepsilon}(t_2, x_2), \left[\frac{\varphi^\varepsilon}{\varepsilon}(t_2, \cdot)\right]\right)(x_2) \cdot |f_x(t_2, x_2)| \geq \frac{\theta}{2}.$$

$\square$

Getting the contradiction. We have that for ε small enough

$$u^\varepsilon + \eta \leq \varphi = g(t) + \overline{p}_- x 1_{\{x<0\}} + \overline{p}_+ x 1_{\{x>0\}} \quad \text{on } Q_{\bar{r},2\bar{r}}(\bar{t}, 0) \backslash Q_{\bar{r},\bar{r}}(\bar{t}, 0)$$
$$\xi^\varepsilon + \eta \leq \varphi = g(t) + \overline{p}_- x 1_{\{x<0\}} + \overline{p}_+ x 1_{\{x>0\}} \quad \text{on } Q_{\bar{r},2\bar{r}}(\bar{t}, 0) \backslash Q_{\bar{r},\bar{r}}(\bar{t}, 0).$$

Using the fact that $w^\varepsilon, \zeta^\varepsilon \to W$ with $W(x) = \bar{p}_- x 1_{\{x<0\}} + \bar{p}_+ x 1_{\{x>0\}}$ (see Theorem 6.1), we deduce that for ε small enough, we have

$$u^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \quad \text{and} \quad \xi^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \quad \text{on } Q_{\bar{r},2\bar{r}}(\bar{t},0) \backslash Q_{\bar{r},\bar{r}}(\bar{t},0).$$

Combining this with (7.15) and (7.16), we get that

$$u^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \quad \text{and} \quad \xi^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \quad \text{outside } Q_{\bar{r},\bar{r}}(\bar{t},0).$$

By the comparison principle on bounded subsets the previous inequality holds in $Q_{\bar{r},\bar{r}}(\bar{t},0)$. Passing to the limit as $\varepsilon \to 0$ and evaluating the inequality in $(\bar{t},0)$, we obtain

$$\overline{u}(\bar{t},0) + \frac{\eta}{2} \leq \varphi(\bar{t},0) \quad \text{and} \quad \overline{\xi}(\bar{t},0) + \frac{\eta}{2} \leq \varphi(\bar{t},0)$$

which is a contradiction with the fact that $\overline{v}(\bar{t},0) = \max\left(\overline{u}(\bar{t},0), \overline{\xi}(\bar{t},0)\right) = \varphi(\bar{t},0)$.

□

8 Proof of the existence of correctors at the junction

This section contains the proof of Theorem 6.1. We proceed as in [13, 14] and we will construct correctors on a truncated domain and then pass to the limit as the size of the domain goes to infinity.

For $l \in (r, +\infty)$, $r << l$ and $r \leq R << l$ we want to find $\lambda_{l,R} \in \mathbb{R}$ such that there exists a solution $(w^{l,R}, \chi^{l,R})$ of

$$\left\{ \begin{array}{ll} \left\{ \begin{array}{l} G^1_R(x, w^{l,R}(x), [\chi^{l,R}], w^{l,R}_x) = \lambda_{l,R} \\ G^2_R(x, \chi^{l,R}(x), [w^{l,R}], \chi^{l,R}_x) = \lambda_{l,R} \end{array} \right. & \text{if } x \in (-l, l) \\ \left\{ \begin{array}{l} \overline{H}^+(w^{l,R}_x) = \lambda_{l,R} \\ \overline{H}^+(\chi^{l,R}_x) = \lambda_{l,R} \end{array} \right. & \text{if } x = l \\ \left\{ \begin{array}{l} \overline{H}^-(w^{l,R}_x) = \lambda_{l,R} \\ \overline{H}^-(\chi^{l,R}_x) = \lambda_{l,R} \end{array} \right. & \text{if } x = -l \end{array} \right. \tag{8.1}$$

with

$$G^1_R(x, w(x), [\chi], q) = \psi_R(x) M(w(x), [\chi])(x)|q| + (1 - \psi_R(x))\overline{H}(q), \tag{8.2}$$

$$G^2_R(x, \chi(x), [w], q) = \psi_R(x) L\left(x, \chi(x), [w]\right)(x)|q| + (1 - \psi_R(x))\overline{H}(q). \tag{8.3}$$

Moreover, $\psi_R \in C^\infty$, $\psi_R : \mathbb{R} \to [0,1]$, with

$$\psi_R \equiv \left\{ \begin{array}{ll} 1 & \text{on } [-R, R] \\ 0 & \text{on } (-\infty, -R-1] \cup [R+1, +\infty), \end{array} \right. \quad \text{and } \psi_R(x) < 1 \ \forall x \notin [-R, R]. \tag{8.4}$$

As in the previous sections, to $G^{1,2}_R$ we associate $\tilde{G}^{1,2}_R$ which is defined in the same way but we replace the non-local operators M and L respectively by $\tilde{M}$ and $\tilde{L}$.

8.1 Comparison principle for a truncated problem

Proposition 8.1 (Comparison principle on a truncated domain)**.** *Let us consider the following problem for $r < l_1 < l_2$ and $\lambda \in \mathbb{R}$, with $l_2 >> R$.*

$$\left\{ \begin{array}{ll} \left\{ \begin{array}{l} \tilde{G}^1_R(x, u(x), [\xi], u_x) \geq \lambda \\ \tilde{G}^2_R(x, \xi(x), [u], \xi_x) \geq \lambda \end{array} \right. & \textit{if } x \in (l_1, l_2) \\ \left\{ \begin{array}{l} \overline{H}^+(u_x) \geq \lambda \\ \overline{H}^+(\xi_x) \geq \lambda \end{array} \right. & \textit{if } x = l_2 \end{array} \right. \tag{8.5}$$

and for $\varepsilon_0 > 0$,

$$\begin{cases} \begin{cases} G^1_R(x, v(x), [\zeta], v_x) \le \lambda - \varepsilon_0 \\ G^2_R(x, \zeta(x), [v], \zeta_x) \le \lambda - \varepsilon_0 \end{cases} & \text{if } x \in (l_1, l_2) \\ \begin{cases} \overline{H}^+(v_x) \le \lambda - \varepsilon_0 \\ \overline{H}^+(\zeta_x) \le \lambda - \varepsilon_0 \end{cases} & \text{if } x = l_2 \end{cases} \tag{8.6}$$

Then if $v(l_1) \le u(l_1)$ *and* $\zeta(l_1) \le \xi(l_1)$, *we have* $v \le u$ *and* $\zeta \le \xi$ *in* $[l_1, l_2]$.

Proof. Like in [13], the only new difficulty to prove this proposition is the comparison at l_2. But since near l_2, the system decouples itself, we can prooceed as in [14, Proposition 4.1].

□

Remark 8.2. *We have a similar result if we exchange the boundary conditions, that is to say for* $l_1 < l_2 < -r$ *and* $l_2 < -R$, *and if the Dirichlet condition is placed in* $x = l_2$ *and the following conditions are imposed at* $x = l_1$,

$$\begin{cases} \begin{cases} \overline{H}^-(u_x) \ge \lambda \\ \overline{H}^-(\xi_x) \ge \lambda \end{cases} & \text{if } x = l_1 \\ \begin{cases} \overline{H}^-(v_x) \le \lambda - \varepsilon_0 \\ \overline{H}^-(\zeta_x) \le \lambda - \varepsilon_0 \end{cases} & \text{if } x = l_1. \end{cases}$$

8.2 Existence of correctors on a truncated domain

Proposition 8.3 (Existence of correctors on a truncated domain)**.** *There exists a constant* $\lambda_{l,R} \in \mathbb{R}$ *such that there exists a solution* $(w^{l,R}, \chi^{l,R})$ *of* (8.1) *for which there exists a constant* C *(depending only on* k_0*) and a Lipschitz continuous function* $m^{l,R}$, *such that*

$$\begin{cases} H_0 \le \lambda_{l,R} \le 0, & \\ |w^{l,R}(x) - m^{l,R}(x)| \le C & \text{for all } x \in [-l, l], \\ |\chi^{l,R}(x) - m^{l,R}(x)| \le C & \text{for all } x \in [-l, l], \\ |m^{l,R}(x) - m^{l,R}(y)| \le C|x-y| & \text{for all } x, y \in [-l, l], \\ |w^{l,R}(x) - \chi^{l,R}(x)| \le C & \text{for all } x \in [-l, l], \end{cases} \tag{8.7}$$

with H_0 *defined in* (2.6).

Proof. Classically, we consider the approximated truncated cell problem,

$$\begin{cases} \begin{cases} \delta v^\delta + G^1_R(x, v^\delta(x), [\zeta^\delta], v^\delta_x) = 0 \\ \delta \zeta^\delta + G^2_R(x, \zeta^\delta(x), [v^\delta], \zeta^\delta_x) = 0 \end{cases} & \text{if } x \in (-l, l) \\ \begin{cases} \delta v^\delta + \overline{H}^+(v^\delta_x) = 0 \\ \delta \zeta^\delta + \overline{H}^+(\zeta^\delta_x) = 0 \end{cases} & \text{if } x = l \\ \begin{cases} \delta v^\delta + \overline{H}^-(v^\delta_x) = 0 \\ \delta \zeta^\delta + \overline{H}^-(\zeta^\delta_x) = 0 \end{cases} & \text{if } x = -l. \end{cases} \tag{8.8}$$

Step 1: construction of barriers. Using that $(0, 0)$ and $(C_0/\delta, C_0/\delta)$ are respectively obvious sub and super-solution of (8.8), with $C_0 = |\min_{p \in \mathbb{R}} \overline{H}_0(p)| = -H_0$ and that we have a comparison principle, we deduce that there exists a continuous viscosity solution (v^δ, ζ^δ) of (8.8) which satisfies

$$0 \le v^\delta \le \frac{C_0}{\delta} \quad \text{and} \quad 0 \le \zeta^\delta \le \frac{C_0}{\delta}. \tag{8.9}$$

Step 2: control of the oscillations of v^δ and ζ^δ.

Lemma 8.4. *The functions v^δ and ζ^δ satisfy for all $x, y \in [-l, l]$, $x \geq y$,*

$$-k_0(x-y) - 1 \leq v^\delta(x) - v^\delta(y) \leq 0 \quad \textit{and} \quad -k_0(x-y) - 1 \leq \zeta^\delta(x) - \zeta^\delta(y) \leq 0. \tag{8.10}$$

Proof of Lemma 8.4. In the rest of the proof we will use the following notation

$$\Omega = \left\{(x, y) \in [-l, l]^2 \text{ s.t. } x \geq y\right\}.$$

Proof of the upper inequality. We want to prove that

$$\overline{M} = \sup_{(x,y)\in\Omega} \max\left(v^\delta(x) - v^\delta(y), \zeta^\delta(x) - \zeta^\delta(y)\right) \leq 0. \tag{8.11}$$

We argue by contradiction and assume that $\overline{M} > 0$. Since v^δ and ζ^δ are continuous and x, y belong to a compact, $\overline{M}$ is reached for a finite point that we denote by $(\bar{x}, \bar{y}) \in \Omega$. Given that $\overline{M} > 0$, we deduce that $\bar{x} \neq \bar{y}$. Therefore, we can use the viscosity inequalities for (8.8).

Let us for instance assume that $\overline{M} = v^\delta(\bar{x}) - v^\delta(\bar{y})$, the other case is similar so we skip it. We distinguish 3 cases:

-If $(\bar{x}, \bar{y}) \in (-l, l)$, we have

$$\begin{aligned} \delta v^\delta(\bar{x}) + G^1_R(\bar{x}, v^\delta(\bar{x}), [\zeta^\delta], 0) \leq 0 \\ \delta v^\delta(\bar{y}) + \tilde{G}^1_R(\bar{y}, v^\delta(\bar{y}), [\zeta^\delta], 0) \geq 0. \end{aligned}$$

Combining these inequalities with the fact that $G^i_R(x, U, [\Xi], 0) = 0$ for $i = 1, 2$, we obtain

$$\delta\overline{M} \leq 0.$$

-If $\bar{x} = l$ and $\bar{y} \in [-l, l)$, we obtain similarly

$$\delta\overline{M} \leq 0, \tag{8.12}$$

using the fact that $\overline{H}^+(0) = 0$.

-If $\bar{x} \in (-l, l]$ and $\bar{y} = -l$, we obtain

$$\delta\overline{M} \leq H_0 < 0,$$

where we have used the fact that $\overline{H}^-(0) = H_0 < 0$.

For every value of $\bar{x}, \bar{y}$ we obtain a contradiction, therefore $\overline{M} \leq 0$.

Proof of the lower inequalities. In order to proof these inequalities, we will use the following lemma which proof is postponed.

Lemma 8.5. *For all $x \in [-l, l]$, we have*

$$0 \leq \zeta^\delta(x) - v^\delta(x) \leq 1. \tag{8.13}$$

In order to prove (8.10), using Lemma 8.5 it is sufficient to prove that

$$\overline{M} = \sup_{(x,y)\in\Omega} \left(\zeta^\delta(y) - v^\delta(x) - k_0(x-y) - 1\right) \leq 0. \tag{8.14}$$

We argue by contradiction and assume that $\overline{M} > 0$. Since Ω is compact and v^δ and ζ^δ are continuous, $\overline{M}$ is reached for a finite point that we denote by $(\bar{x}, \bar{y}) \in \Omega$. Since $\overline{M} > 0$, we deduce that $\bar{x} > \bar{y}$ (thanks to Lemma 8.5). Therefore, we can use the viscosity inequalities for (8.8). We distinguish 4 cases:

-If $\bar{x}, \bar{y} \in (-l, l)$, we obtain

$$\begin{array}{l} \delta\zeta^\delta(\bar{y}) + G_R^2(\bar{y}, \zeta^\delta(\bar{y}), [v^\delta], -k_0) \leq 0 \\ \delta v^\delta(\bar{x}) + \tilde{G}_R^1(\bar{x}, v^\delta(\bar{x}), [\zeta^\delta], -k_0) \geq 0, \end{array}$$

combining these inequalities and using the definition of $\overline{M}$, we obtain

$$\delta\overline{M} \leq \delta\zeta^\delta(\bar{y}) - \delta v^\delta(\bar{x}) \leq \begin{array}{l} \tilde{G}_R^1(\bar{x}, v^\delta(\bar{x}), [\zeta^\delta], -k_0) \\ -G_R^2(\bar{y}, \zeta^\delta(\bar{y}), [v^\delta], -k_0). \end{array} \tag{8.15}$$

Since the non-local operator $\tilde{M}$ is negative and that $\overline{H}(-k_0) = 0$ we deduce that

$$\tilde{G}_R^1(\bar{x}, v^\delta(\bar{x}), [\zeta^\delta], -k_0) \leq 0.$$

We now claim that $G_R^2(\bar{y}, \zeta^\delta(\bar{y}), [v^\delta], -k_0) \geq 0$. Using $\overline{H}(-k_0) = 0$ and (3.15), we get that

$$\begin{aligned} G_R^2(\bar{y}, \zeta^\delta(\bar{y}), [v^\delta], -k_0) = &\quad L\left(\bar{y}, \zeta^\delta(\bar{y}), \left[v^\delta(\cdot)\right]\right)(\bar{y}) \cdot k_0\psi_R(\bar{y}) \\ \geq &\quad -2k_0 V\left(N\left(\zeta^\delta(\bar{y}), \left[v^\delta(\cdot)\right]\right)(\bar{y})\right). \end{aligned} \tag{8.16}$$

Let us now prove that $N(\zeta^\delta(\bar{y}), [v^\delta(\cdot)])(\bar{y}) \leq h_0$. In fact, it is sufficient to prove that for all $z \in (h_0, D]$, we have

$$v^\delta(\bar{y} + z) - \zeta^\delta(\bar{y}) < -1. \tag{8.17}$$

First, if $z \geq \bar{x} - \bar{y}$, using the fact that v^δ is non increasing and that $\overline{M} > 0$, we obtain

$$v^\delta(\bar{y} + z) - \zeta^\delta(\bar{y}) \leq v^\delta(\bar{x}) - \zeta^\delta(\bar{y}) \leq -k_0(\bar{x} - \bar{y}) - 1 < -1.$$

Second, in the case $z < \bar{x} - \bar{y}$, using the fact that

$$\zeta^\delta(\bar{y} + z) - v^\delta(\bar{x}) - k_0(\bar{x} - \bar{y} - z) - 1 \leq \zeta^\delta(\bar{y}) - v^\delta(\bar{x}) - k_0(\bar{x} - \bar{y}) - 1,$$

and using Lemma 8.5 we deduce that

$$v^\delta(\bar{y} + z) - \zeta^\delta(\bar{y}) \leq -k_0 z < -1. \tag{8.18}$$

This implies that $N(\zeta^\delta(\bar{y}), [v^\delta(\cdot)])(\bar{y}) \leq h_0$. Using assumption (A3) ($V(h = 0)$ if $h \leq h_0$) and injecting this result in (8.16) we get that $G_R^2(\bar{y}, \zeta^\delta(\bar{y}), [v^\delta], -k_0) \geq 0$. Using (8.15) we then get a contradiction.

-If $\bar{x} \in (-l, l)$ and $\bar{y} = -l$, we obtain

$$\begin{array}{l} \delta\zeta^\delta(\bar{y}) + \overline{H}^-(-k_0) \leq 0 \\ \delta v^\delta(\bar{x}) + \tilde{G}_R^1(\bar{x}, v^\delta(\bar{x}), [\zeta^\delta], -k_0) \geq 0. \end{array}$$

Using the fact that $\overline{H}^-(-k_0) = 0$ and that $\tilde{G}_R^1(\bar{x}, v^\delta(\bar{x}), [\zeta^\delta], -k_0) \leq 0$ we obtain $\delta\overline{M} \leq 0$.

-If $\bar{x} = l$ and $\bar{y} \in (-l, l)$, we obtain

$$\begin{array}{l} \delta\zeta^\delta(\bar{y}) + G_R^2(\bar{y}, \zeta^\delta(\bar{y}), [v^\delta], -k_0) \leq 0 \\ \delta v^\delta(\bar{x}) + \overline{H}^+(-k_0) \geq 0, \end{array}$$

using that $G_R^2(\bar{y}, \zeta^\delta(\bar{y}), [v^\delta], -k_0) \geq 0$ (see the first case) , and the fact that $\overline{H}^+(-k_0) < 0$, we directly obtain $\delta\overline{M} \leq 0$.

-If $\bar{x} = l$ and $\bar{y} = -l$, we obtain

$$\begin{array}{l} \delta\zeta^\delta(\bar{y}) + \overline{H}^-(-k_0) \leq 0 \\ \delta v^\delta(\bar{x}) + \overline{H}^+(-k_0) \geq 0, \end{array}$$

and so, we get $\delta\overline{M} \leq 0$.

For every value of $\bar{x}, \bar{y} \in [-l, l]$ we get a contradiction, therefore we have $\overline{M} \leq 0$. This ends the proof of Lemma 8.4.

□

Step 3: construction of a Lipschitz estimate. We want to construct a Lipschitz continuous function m^δ, such that there exists a constant $C > 0$ (independent of l and R) such that

$$\begin{cases} |v^\delta(x) - m^\delta(x)| \le C & \text{for all } x \in [-l, l], \\ |\zeta^\delta(x) - m^\delta(x)| \le C & \text{for all } x \in [-l, l], \\ |m^\delta(x) - m^\delta(y)| \le C|x-y| & \text{for all } x, y \in [-l, l]. \end{cases} \tag{8.19}$$

We define m^δ as an affine function in each interval of the form $[ih_0, (i+1)h_0]$, with $i \in \mathbb{Z}$, such that

$$m^\delta(ih_0) = v^\delta(ih_0) \quad \text{and} \quad m^\delta((i+1)h_0) = v^\delta((i+1)h_0).$$

Since m^δ and v^δ are non-increasing, and $|v^\delta((i+1)h_0) - v^\delta(ih_0)| \le k_0 h_0 + 1 = 2$, we deduce that for all $x \in [ih_0, (i+1)h_0]$,

$$-2 \le v^\delta((i+1)h_0) - m^\delta(ih_0) \le v^\delta(x) - m^\delta(x) \le v^\delta(ih_0) - m^\delta((i+1)h_0) \le 2, \tag{8.20}$$

and for all $x, y \in [-l, l]$,

$$|m^\delta(x) - m^\delta(y)| \le 2k_0|x-y|.$$

Now using Lemma 8.5, we have

$$|\zeta^\delta(x) - m^\delta(x)| \le 3.$$

Choosing $C = \max(2k_0, 3)$, we obtain (8.19).

Step 4: passing to the limit as δ goes to 0. Using (8.9), Lemma 8.5 and (8.19), we deduce that there exists a subsequence $\delta_n \to 0$ such that

$$\begin{array}{ll} \delta_n v^{\delta_n}(0) \to -\lambda_{l,R} & \text{as } n \to +\infty, \\ \delta_n \zeta^{\delta_n}(0) \to -\lambda_{l,R} & \text{as } n \to +\infty, \\ m^{\delta_n} - m^{\delta_n}(0) \to m^{l,R} & \text{as } n \to +\infty. \end{array}$$

The last convergence being locally uniform. Let us consider,

$$\overline{w}^{l,R} = \limsup_{\delta_n \to 0}{}^* (v^{\delta_n} - v^{\delta_n}(0)) \quad \text{and} \quad \underline{w}^{l,R} = \liminf_{\delta_n \to 0}{}_* (v^{\delta_n} - v^{\delta_n}(0))$$

and

$$\overline{\chi}^{l,R} = \limsup_{\delta_n \to 0}{}^* (\zeta^{\delta_n} - \zeta^{\delta_n}(0)) \quad \text{and} \quad \underline{\chi}^{l,R} = \liminf_{\delta_n \to 0}{}_* (\zeta^{\delta_n} - \zeta^{\delta_n}(0)).$$

Therefore, we have that $\lambda_{l,R}$, $\overline{w}^{l,R}$, $\underline{w}^{l,R}$, $\overline{\chi}^{l,R}$, $\underline{\chi}^{l,R}$ and $m^{l,R}$ satisfy

$$\begin{array}{r} H_0 \le \lambda_{l,R} \le 0, \\ |\overline{w}^{l,R} - m^{l,R}| \le C, \ |\underline{w}^{l,R} - m^{l,R}| \le C, \\ |\overline{\chi}^{l,R} - m^{l,R}| \le C, \ |\underline{\chi}^{l,R} - m^{l,R}| \le C, \\ |m_x^{l,R}| \le C, \end{array} \tag{8.21}$$

and thanks to Lemma 8.5, we have

$$|\underline{\chi}^{l,R} - \overline{w}^{l,R}|, \ |\overline{\chi}^{l,R} - \underline{w}^{l,R}| \le 1. \tag{8.22}$$

By stability of viscosity solutions, we have that $(\overline{w}^{l,R} - 2C, \overline{\chi}^{l,R} - 2C)$ and $(\underline{w}^{l,R}, \underline{\chi}^{l,R})$ are respectively a sub-solution and a super-solution of (8.1), and

$$\overline{w}^{l,R} - 2C \le \underline{w}^{l,R} \quad \text{and} \quad \overline{\chi}^{l,R} - 2C \le \underline{\chi}^{l,R}.$$

By Perron's method, we can construct a solution $(w^{l,R}, \chi^{l,R})$ of (8.1) and thanks to (8.21) and (8.22), $m^{l,R}$, $w^{l,R}$, $\chi^{l,R}$ and $\lambda_{l,R}$ satisfy (8.7).

The uniqueness of $\lambda_{l,R}$ is classical so we skip it. This ends the proof of Proposition 8.3.

□

Proof of Lemma 8.5. We separate the proof in two parts. This proof uses the vertex test function of the work of Imbert and Monneau [19, Theorem 3.2] to treat the comparison between v^δ and ζ^δ near $-l$ and l. In fact, we consider that we have a network composed of a single branch with two nodes (one in $-l$ and the other in l). Near $-l$ we consider an outgoing branch and near l we consider an incoming branch.

Step 1: proof of $v^\delta(x) - \zeta^\delta(x) \leq 0$ for all $x \in [-l, l]$. We want to prove that

$$\overline{M} = \sup_{x\in[-l,l]} \left(v^\delta(x) - \zeta^\delta(x)\right) \leq 0.$$

We argue by contradiction and assume that $\overline{M} > 0$. Given that v^δ and ζ^δ are continuous, $\overline{M}$ is reached at a finite point that we denote by $\bar{x} \in [-l, l]$. We distinguish 3 cases according to the position of $\bar{x}$ in the interval $[-l, l]$.

Case 1: $\bar{x} \in (-l, l)$. We define for ε a small parameter,

$$\varphi(x, y) = v^\delta(x) - \zeta^\delta(y) - \frac{(x-y)^2}{2\varepsilon} - \frac{1}{2}\left((x-\bar{x})^2 + (y-\bar{x})^2\right).$$

Since $[-l, l]$ is compact and v^δ and ζ^δ are continuous functions, the function φ reaches a maximum at a finite point that we denote by $(x_\varepsilon, y_\varepsilon) \in [-l, l]$. If we denote $M_\varepsilon = \varphi(x_\varepsilon, y_\varepsilon)$, by classical arguments, we have that

$$\lim_{\varepsilon\to 0} M_\varepsilon = M, \quad \lim_{\varepsilon\to 0} |x_\varepsilon - y_\varepsilon| = 0, \quad \text{and} \quad (x_\varepsilon, y_\varepsilon) \to (\bar{x}, \bar{x}) \text{ as } \varepsilon \text{ goes to } 0. \tag{8.23}$$

We can also prove that

$$\frac{(x_\varepsilon - y_\varepsilon)^2}{\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0. \tag{8.24}$$

Furthermore, for ε small enough we have $x_\varepsilon, y_\varepsilon \in (-l, l)$, and using the viscosity inequalities we obtain

$$\begin{aligned} &\delta v^\delta(x_\varepsilon) + G^1_R(x_\varepsilon, v^\delta(x_\varepsilon), [\zeta^\delta], p_\varepsilon + (x_\varepsilon - \bar{x})) \leq 0 \\ &\delta\zeta^\delta(y_\varepsilon) + \tilde{G}^2_R(y_\varepsilon, \zeta^\delta(y_\varepsilon), [v^\delta], p_\varepsilon - (y_\varepsilon - \bar{x})) \geq 0, \end{aligned}$$

with $p_\varepsilon = (x_\varepsilon - y_\varepsilon)/\varepsilon$. Combining these inequalities and using the definition of $\overline{M}$, we obtain that

$$\begin{aligned} \delta\overline{M} \leq\ & \tilde{G}^2_R\left(y_\varepsilon, \zeta^\delta(y_\varepsilon), [v^\delta], p_\varepsilon - (y_\varepsilon - \bar{x})\right) - G^1_R\left(x_\varepsilon, v^\delta(x_\varepsilon), [\zeta^\delta], p_\varepsilon + (x_\varepsilon - \bar{x})\right) \\ \leq\ & (\psi_R(x_\varepsilon) - \psi_R(y_\varepsilon))\overline{H}(p_\varepsilon) + ||\psi_R||_\infty ||\overline{H}'||_\infty \left(|y_\varepsilon - \bar{x}| + |x_\varepsilon - \bar{x}|\right) \\ & + \psi_R(y_\varepsilon)\alpha\tilde{K}(\zeta^\delta(y_\varepsilon), [v^\delta(\cdot)])(y_\varepsilon).|p_\varepsilon - y_\varepsilon + \bar{x}| \\ & - \psi_R(x_\varepsilon)M(v^\delta(x_\varepsilon), [\zeta^\delta(\cdot)])(x_\varepsilon).|p_\varepsilon + x_\varepsilon - \bar{x}| \\ \leq\ & (\psi_R(x_\varepsilon) - \psi_R(y_\varepsilon))\overline{H}(p_\varepsilon) + ||\psi_R||_\infty ||\overline{H}'||_\infty \left(|y_\varepsilon - \bar{x}| + |x_\varepsilon - \bar{x}|\right) \\ & + \psi_R(y_\varepsilon)\alpha\tilde{K}(\zeta^\delta(y_\varepsilon), [v^\delta(\cdot)])(y_\varepsilon).|p_\varepsilon| - \psi_R(x_\varepsilon)M(v^\delta(x_\varepsilon), [\zeta^\delta(\cdot)])(x_\varepsilon).|p_\varepsilon| \\ & + (\alpha M_0|y_\varepsilon - \bar{x}| + M_0|x_\varepsilon - \bar{x}|) \\ \leq\ & (\psi_R(x_\varepsilon) - \psi_R(y_\varepsilon))\overline{H}(p_\varepsilon) + o_\varepsilon(1) \\ & + \psi_R(y_\varepsilon)\alpha\tilde{K}(\zeta^\delta(y_\varepsilon), [v^\delta(\cdot)])(y_\varepsilon).|p_\varepsilon| - \psi_R(x_\varepsilon)M(v^\delta(x_\varepsilon), [\zeta^\delta(\cdot)])(x_\varepsilon).|p_\varepsilon| \end{aligned} \tag{8.25}$$

where we have replaced G^1_R and $\tilde{G}^2_R$ by their definitions, used the fact that by definition $\overline{H}$ is a Lipschitz function and that that $V \geq 0$ for the second inequality, used Remark 3.5 for the third inequality and (8.23) for the last inequality.

We will compute the right part of the inequality in different steps.

1-Concerning the local operator.

$$\begin{aligned} \left|(\psi_R(x_\varepsilon) - \psi_R(y_\varepsilon))\overline{H}(p_\varepsilon)\right| &\leq ||D\psi_R||_\infty |x_\varepsilon - y_\varepsilon||\overline{H}(p_\varepsilon)| \\ &\leq ||D\psi_R||_\infty V_{max}\frac{(x_\varepsilon - y_\varepsilon)^2}{\varepsilon} \\ &= o_\varepsilon(1) \end{aligned} \tag{8.26}$$

where we have used the regularity of ψ_R for the first inequality, used the fact that by definition of $\overline{H}$, we have $|\overline{H}| \le V_{max}|p|$ for the second inequality and used (8.24) for the last inequality.

2-Concerning the non-local operator M. We claim that $M(v^\delta(x_\varepsilon), [\zeta^\delta(\cdot)])(x_\varepsilon) \le |x_\varepsilon - y_\varepsilon|$. To prove this, it suffices to prove that for all $z > |x_\varepsilon - y_\varepsilon|$

$$\zeta^\delta(x_\varepsilon + z) - v^\delta(x_\varepsilon) < 0.$$

Using the fact that ζ^δ is decreasing, that $x_\varepsilon + z \ge y_\varepsilon$ and that $M_\varepsilon > 0$, we obtain

$$\zeta^\delta(x_\varepsilon + z) - v^\delta(x_\varepsilon) \le \zeta^\delta(y_\varepsilon) - v^\delta(x_\varepsilon) < 0.$$

Therefore we have

$$-\psi_R(x_\varepsilon)M(v^\delta(x_\varepsilon), [\zeta^\delta])(x_\varepsilon) = -\psi_R(x_\varepsilon)\int_0^{|x_\varepsilon - y_\varepsilon|} E(\zeta^\delta(x_\varepsilon + z) - v^\delta(x_\varepsilon))dz \le \alpha|x_\varepsilon - y_\varepsilon|. \quad (8.27)$$

In particular, this implies that

$$\Big|\psi_R(x_\varepsilon)M(v^\delta(x_\varepsilon), [\zeta^\delta])(x_\varepsilon)\Big||p_\varepsilon| \le \alpha\frac{(x_\varepsilon - y_\varepsilon)^2}{\varepsilon} = o_\varepsilon(1). \quad (8.28)$$

3-Concerning the non-local operator $\tilde{K}$. We claim that $|\tilde{K}(\zeta\delta(y_\varepsilon), [v^\delta(\cdot)])(y_\varepsilon)| \le |x_\varepsilon - y_\varepsilon|$. As before, it suffices to prove that for all $z > |x_\varepsilon - y_\varepsilon|$

$$v^\delta(y_\varepsilon - z) - \zeta^\delta(y_\varepsilon) > 0.$$

Using the fact that v^δ is decreasing, that $x_\varepsilon \ge y_\varepsilon - z$ and that $M_\varepsilon > 0$, we obtain

$$v^\delta(y_\varepsilon - z) - \zeta^\delta(y_\varepsilon) \ge v^\delta(x_\varepsilon) - \zeta^\delta(y_\varepsilon) > 0.$$

Therefore we have

$$\Big|\psi_R(y_\varepsilon)\tilde{K}(\zeta^\delta(y_\varepsilon), [v^\delta])(y_\varepsilon)\Big| \le |x_\varepsilon - y_\varepsilon|. \quad (8.29)$$

Injecting (8.26), (8.27), and (8.29) into (8.25), we obtain $\delta\overline{M} \le o_\varepsilon(1)$ and we get a contradiction for ε small enough.

Case 2: $\bar{x} = l$. In this case, we use the vertex test function introduced by Imbert and Monneau. We refer to [19] for a detailed description of the vertex test function, but for the readers convenience we recall the properties that we used to complete this proof. The vertex test function G^γ is associated to the single Hamiltonian $\overline{H}$. We fix $\gamma = \delta M/2$. It satisfies the following properties.

1. (Regularity)

$$G^\gamma \in C([-l,l]^2) \quad \begin{cases} G^\gamma(x,\cdot) \in C^1([-l,l]) & \text{for all} \quad x \in [-l,l] \\ G^\gamma(\cdot,y) \in C^1([-l,l]) & \text{for all} \quad y \in [-l,l]. \end{cases} \quad (8.30)$$

2. (Bound from below) $G^\gamma \ge 0 = G(0,0)$.

3. (Super-linearity) There exists $g : [0,+\infty) \to \mathbb{R}$ non-decreasing and such that for all $(x,y) \in [-l,l]^2$

$$g(|x-y|) \le G^\gamma(x,y) \quad \text{and} \quad \lim_{a\to+\infty}\frac{g(a)}{a} = +\infty.$$

4. (Compatibility condition on the gradient)

$$\overline{H}(y, -G_y^\gamma(x,y)) - \overline{H}(x, G_x^\gamma(x,y)) \leq \gamma, \tag{8.31}$$

with for all $x \in [-l, l]$ and $p \in \mathbb{R}$,

$$\overline{H}(x,p) = \begin{cases} \overline{H}(p) & \text{if } x \in [-l, l) \\ \overline{H}^+(p) & \text{if } x = l. \end{cases} \tag{8.32}$$

We introduce the following test function, for $\varepsilon > 0$ a small parameter,

$$\varphi(x,y) = v^\delta(x) - \zeta^\delta(y) - \varepsilon G^\gamma\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) - \frac{1}{2}\left((x-\bar{x})^2 + (y-\bar{x})^2\right).$$

which like before reaches a maximum at a finite point $(x_\varepsilon, y_\varepsilon) \in [-l, l]$ and (8.23) remains true.

Using the viscosity equations, we have that

$$\begin{cases} \delta v^\delta(x_\varepsilon) + \overline{H}\left(x_\varepsilon, G_x^\gamma\left(\frac{x_\varepsilon}{\varepsilon}, \frac{y_\varepsilon}{\varepsilon}\right) + (x_\varepsilon - \bar{x})\right) \leq 0 \\ \delta\zeta^\delta(y_\varepsilon) + \overline{H}\left(y_\varepsilon, -G_y^\gamma\left(\frac{x_\varepsilon}{\varepsilon}, \frac{y_\varepsilon}{\varepsilon} - (y_\varepsilon - \bar{y})\right)\right) \geq 0. \end{cases}$$

Using the definition of $\overline{M}$ and combining the previous inequalities, we get that

$$\begin{aligned} \delta\overline{M} \leq& \overline{H}\left(y_\varepsilon, -G_y^\gamma\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon} - (y_\varepsilon - \bar{y})\right)\right) - \overline{H}\left(x_\varepsilon, G_x^\gamma\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) + (x_\varepsilon - \bar{x})\right) \\ \leq& \overline{H}\left(y_\varepsilon, -G_y^\gamma\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)\right) - \overline{H}\left(x_\varepsilon, G_x^\gamma\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)\right) + o_\varepsilon(1), \end{aligned}$$

where we have used (8.23) combined with the fact that both $\overline{H}$ and $\overline{H}^+$ are Lipschitz continuous for the second inequality. Using the compatibility condition on the gradient of the vertex test function (8.31) we obtain

$$\delta\overline{M} \leq \gamma + o_\varepsilon(1),$$

and given that $\gamma = \delta\overline{M}/2$, we get a contradiction for ε small enough.

Case 3: $\bar{x} = -l$. This case is exactly like the previous one with the exception that the vertex test function must be adapted to treat the junction at $-l$. In particular, (8.32) is replaced by

$$\overline{H}(x,p) = \begin{cases} \overline{H}(p) & \text{if } x \in (-l, l] \\ \overline{H}^-(p) & \text{if } x = -l. \end{cases}$$

We skip the rest of the computation for this case.

In conclusion, we have $\overline{M} \leq 0$ and for all $x \in [-l, l]$, $0 \leq \zeta^\delta(x) - v^\delta(x)$.

Step 2: proof of $\zeta^\delta(x) - v^\delta(x) \leq 1$. We want to prove that

$$\overline{M} = \sup_{x \in [-l,l]} \left(\zeta^\delta(x) - v^\delta(x) - 1\right) \leq 0.$$

We argue by contradiction and assume that $\overline{M} > 0$. Give that v^δ and ζ^δ are continuous, $\overline{M}$ is reached at a finite point that we denote by $\bar{x} \in [-l, l]$. We distinguish 2 cases according to the position of $\bar{x}$ in the interval $[-l, l]$.

Case 1: $\bar{x} \in (-l, l)$**.** We define for ε a small parameter,

$$\varphi(x,y) = v^\delta(x) - \zeta^\delta(y) - 1 - \frac{(x-y)^2}{2\varepsilon} - \frac{1}{2}\left((x-\bar{x})^2 + (y-\bar{x})^2\right).$$

Using the same arguments as before, the test function reaches a maximum at a finite point that we denote by $(x_\varepsilon, y_\varepsilon) \in [-l, l]$. If we denote $M_\varepsilon = \varphi(x_\varepsilon, y_\varepsilon)$ (8.23) and (8.24) remain valid.

For ε small enough we have $x_\varepsilon, y_\varepsilon \in (-l, l)$, and using the viscosity inequalities we get that

$$\begin{array}{l} \delta\zeta^\delta(x_\varepsilon) + G^2_R(x_\varepsilon, \zeta^\delta(x_\varepsilon), [v^\delta], p_\varepsilon) \leq 0 \\ \delta v^\delta(y_\varepsilon) + \tilde{G}^1_R(y_\varepsilon, v^\delta(y_\varepsilon), [\zeta^\delta], p_\varepsilon) \geq 0, \end{array}$$

with $p_\varepsilon = (x_\varepsilon - y_\varepsilon)/\varepsilon$. Combining these inequalities and using the definition of $\overline{M}$, we obtain

$$\begin{array}{rl} \delta\overline{M} \leq & \tilde{G}^1_R(y_\varepsilon, v^\delta(y_\varepsilon), [\zeta^\delta], p_\varepsilon) - G^2_R(x_\varepsilon, \zeta^\delta(x_\varepsilon), [v^\delta], p_\varepsilon) \\ \leq & (\psi_R(x_\varepsilon) - \psi_R(y_\varepsilon))\overline{H}(p_\varepsilon) + 2\psi_R(x_\varepsilon) V\left(N(\zeta^\delta(x_\varepsilon), [v^\delta(\cdot)])(x_\varepsilon)\right).|p_\varepsilon|, \end{array} \tag{8.33}$$

where we have replaced G^2_R and $\tilde{G}^1_R$ by their definition and used (3.15) and that $\tilde{M} \leq 0$. We will compute the right part of (8.33) in different steps.

1-Concerning the local operator. Like before, we have

$$\left|(\psi_R(x_\varepsilon) - \psi_R(y_\varepsilon))\overline{H}(p_\varepsilon)\right| \leq ||D\psi_R||_\infty |x_\varepsilon - y_\varepsilon||\overline{H}(p_\varepsilon)| = o_\varepsilon(1). \tag{8.34}$$

2-Concerning the non-local operator N**.** We claim that

$$N(\zeta^\delta(x_\varepsilon), [v^\delta(\cdot)])(x_\varepsilon) \leq h_0.$$

To prove this, it suffices to prove that for all $z \geq h_0$, we have

$$v^\delta(x_\varepsilon + z) - \zeta^\delta(x_\varepsilon) < -1.$$

Since $|x_\varepsilon - y_\varepsilon| \to 0$ as ε goes to 0, we have for all $z \geq h_0$ and ε small enough that $x_\varepsilon + z \geq y_\varepsilon$. Therefore, we get

$$v^\delta(x_\varepsilon + z) - \zeta^\delta(x_\varepsilon) \leq v^\delta(y_\varepsilon) - \zeta^\delta(x_\varepsilon) < -1,$$

where we have used the fact that v^δ is decreasing for the first inequality and the fact that $M_\varepsilon > 0$ for the second inequality. This implies that

$$V\left(N(\zeta^\delta(x_\varepsilon), [v^\delta(\cdot)])(x_\varepsilon)\right) \leq V(h_0) = 0. \tag{8.35}$$

Injecting (8.34) and (8.35) in (8.33), we obtain $\delta\overline{M} \leq o_\varepsilon(1)$, and we get a contradiction for ε small enough.

Case 2: $\bar{x} = l$ **of** $\bar{x} = -l$**.** Proceeding like in the previous step we obtain directly a contradiction by using the properties of the vertex test function.

This ends the proof of Lemma 8.5.

□

Proposition 8.6 (First definition of the flux limiter)**.** *The following limits exists (up to some sub-sequence),*

$$\begin{cases} \overline{A} = \lim\limits_{R\to+\infty} \overline{A}_R, \\ \overline{A}_R = \lim\limits_{l\to+\infty} \lambda_{R,l}. \end{cases}$$

Moreover, we have

$$H_0 \leq \overline{A},\ \overline{A}_R \leq 0. \tag{8.36}$$

Proof. This proposition is a direct consequence of (8.7). □

Proposition 8.7 (Control of the slopes on a truncated domain)**.** *Assume that l and R are big enough. Let $(w^{l,R},\chi^{l,R})$ be the solution of* (8.1) *given by Proposition 8.3. We also assume up to a sub-sequence, that $\overline{A}=\lim_{R\to+\infty}\lim_{l\to+\infty}\lambda_{l,R}>H_0$. Then there exists $\gamma_0>0$ and a constant $C>0$ (independent of l and R) such that for all $\gamma\in(0,\gamma_0)$ and for all $x\geq r+D$, $h\geq 0$ we have*

$$w^{l,R}(x+h)-w^{l,R}(x)\geq(\overline{p}_+-\gamma)h-C \tag{8.37}$$

and

$$\chi^{l,R}(x+h)-\chi^{l,R}(x)\geq(\overline{p}_+-\gamma)h-C. \tag{8.38}$$

Similarly, for all $x\leq -r-D$ and $h\geq 0$,

$$w^{l,R}(x-h)-w^{l,R}(x)\geq(-\overline{p}_--\gamma)h-C \tag{8.39}$$

and

$$\chi^{l,R}(x-h)-\chi^{l,R}(x)\geq(-\overline{p}_--\gamma)h-C. \tag{8.40}$$

Proof. We only do the proof of (8.37)-(8.38), since the proof of (8.39)-(8.40) is similar and we skip it. For $\mu>0$, small enough, we denote by p^+_μ the real number defined by

$$\overline{H}(p^+_\mu)=\overline{H}^+(p^+_\mu)=\lambda_{l,R}-\mu. \tag{8.41}$$

Using that

$$H_0<\lambda_{l,R}\leq 0,$$

we deduce that p^+_μ exists for μ small enough and $p^+_\mu\in[-k_0,0)$.

Let us now consider

$$\begin{cases} w^+=p^+_\mu x,\\ \chi^+=p^+_\mu x-\dfrac{p^+_\mu}{\alpha}V\left(\dfrac{-1}{p^+_\mu}\right),\end{cases}$$

that satisfy

$$\overline{H}(w^+_x)=\overline{H}^+(w^+_x)=\overline{H}(\chi^+_x)=\overline{H}^+(\chi^+_x)=\lambda_{l,R}-\mu\quad\text{for } x\in\mathbb{R}. \tag{8.42}$$

Let us consider $(w,\chi)=\left(0,-\dfrac{p^+_\mu}{\alpha}V\left(\dfrac{-1}{p^+_\mu}\right)\right)$ the correctors provided by Proposition 5.1 for $p=p^+_\mu$. Given the definition of w^+ and χ^+, we get

$$M(w^+(x),[\chi^+])(x)=M_{p^+_\mu}(w(x),[\chi])(x),\quad K(\chi^+(x),[w^+])(x)=K_{p^+_\mu}(\chi(x),[w])(x),$$

and

$$N(\chi^+(x),[w^+])(x)=N_{p^+_\mu}(\chi(x),[w])(x).$$

In particular this implies that

$$M(w^+(x),[\chi^+])(x)=-V\left(\frac{-1}{p^+_\mu}\right)$$

and

$$\alpha K(\chi^+(x),[w^+])(x)-2V\Big(K(\chi^+(x),[w^+])(x)+N(\chi^+(x),[w^+])(x)\Big)=-V\left(\frac{-1}{p^+_\mu}\right).$$

Finally, given that the non-local operator K is bounded by D (see Remark 3.5), we have for all $x \in (r+D, l]$

$$\phi\Big(x - K(\chi^+(x), [w^+])(x)\Big) = 1.$$

Combining the previous results, we can see that the restriction of (w^+, χ^+) to $(r+D, l]$ satisfies

$$\left\{\begin{array}{ll} \left\{\begin{array}{l} G^1_R(x, w^+(x), [\chi^+], w^+_x) = \overline{H}(p^+_\mu) = \lambda_{l,R} - \mu \\ G^2_R(x, \chi^+(x), [w^+], \chi^+_x) = \overline{H}(p^+_\mu) = \lambda_{l,R} - \mu \end{array}\right. & \text{if } x \in (r+D, l) \\ \left\{\begin{array}{l} \overline{H}^+(w^+_x) = \lambda_{l,R} - \mu \\ \overline{H}^+(\chi^+_x) = \lambda_{l,R} - \mu \end{array}\right. & \text{if } x = l \end{array}\right. \tag{8.43}$$

Let us introduce, for some $x_0 \in (r+D, l]$,

$$\left\{\begin{array}{l} g = w^{l,R} - w^{l,R}(x_0) \\ h = \chi^{l,R} - w^{l,R}(x_0), \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l} u = w^+ - w^+(x_0) - C - \dfrac{k_0}{\alpha} V_{max} \\ v = \chi^+ - w^+(x_0) - C - \dfrac{k_0}{\alpha} V_{max}, \end{array}\right. \tag{8.44}$$

with $C > 0$ the constant provided by Proposition 8.3. Then we have

$$g(x_0) = 0 \geq -C - \frac{k_0}{\alpha} V_{max} = u(x_0)$$

and

$$h(x_0) = \chi^{l,R}(x_0) - w^{l,R}(x_0) \geq -C \geq -C - \frac{k_0}{\alpha} V_{max} - \frac{p^+_\mu}{\alpha} V\left(\frac{-1}{p}\right) = v(x_0),$$

where we have used the fact that $p^+_\mu \in [-k_0, 0)$ and $||V||_\infty \leq V_{max}$. Using that (g, h) is a solution of (8.5) and (u, v) is a solution of (8.6) (with $\varepsilon_0 = \mu$), joint to the comparison principle (Proposition 8.1), up to changing the value of the constant C, we get that

$$\left\{\begin{array}{l} w^{l,R}(x) - w^{l,R}(x_0) \geq p^+_\mu (x - x_0) - C \\ \chi^{l,R}(x) - \chi^{l,R}(x_0) \geq p^+_\mu (x - x_0) - C. \end{array}\right.$$

This implies that for all $h \geq 0$, and for all $x \in (r+D, l)$,

$$\left\{\begin{array}{l} w^{l,R}(x+h) - w^{l,R}(x) \geq p^+_\mu h - C \\ \chi^{l,R}(x+h) - \chi^{l,R}(x) \geq p^+_\mu h - C. \end{array}\right.$$

Finally, if we choose $\gamma_0 < |p_0 - \overline{p}_+|$, then we have

$$\overline{H}(\overline{p}_+ - \gamma) = \overline{H}^+(\overline{p}_+ - \gamma).$$

Choosing $\mu > 0$ such that

$$p^+_\mu = \overline{p}_+ - \gamma.$$

we obtain (8.37)-(8.38).

□

Proof of Theorem 6.1. The proof is performed in two steps.

Step 1: proof of i) and ii) We want to pass to the limit as $l \to +\infty$ and then as $R \to +\infty$ on the solution of (8.1) given by Proposition 8.3. Using (8.3), there exists $l_n \to +\infty$, such that

$$m^{l_n,R} - m^{l_n,R}(0) \to m^R \quad \text{as } n \to +\infty,$$

the convergence being locally uniform. We also define

$$\overline{w}^R(x) = \limsup_{n\to+\infty}{}^* \left(w^{l_n,R} - w^{l_n,R}(0)\right), \quad \underline{w}^R(x) = \liminf_{n\to+\infty}{}_* \left(w^{l_n,R} - w^{l_n,R}(0)\right),$$

and

$$\overline{\chi}^R(x) = \limsup_{n\to+\infty}{}^* \left(\chi^{l_n,R} - \chi^{l_n,R}(0)\right), \quad \underline{\chi}^R(x) = \liminf_{n\to+\infty}{}_* \left(\chi^{l_n,R} - \chi^{l_n,R}(0)\right)$$

Thanks to (8.3), we know that these limits are finite and satisfy

$$m^R - C \le \underline{w}^R \le \overline{w}^R \le m^R + C. \quad \text{and} \quad m^R - C \le \underline{\chi}^R \le \overline{\chi}^R \le m^R + C.$$

By stability of viscosity solutions $(\overline{w}^R - 2C, \overline{\chi}^R - 2C)$ and $(\underline{w}^R, \underline{\chi}^R)$ are respectively a sub-solution and a super-solution of

$$\begin{cases} G^1_R(x, w^R(x), [\chi^R], w^R_x) = \overline{A}_R \\ G^2_R(x, \chi^R(x), [w^R], \chi^R_x) = \overline{A}_R. \end{cases} \tag{8.45}$$

Therefore, using Perron's method, we can construct a solution (w^R, χ^R) of (8.45), with m^R, $\overline{A}_R$, w^R and χ^R satisfying

$$\begin{cases} |m^R(x) - m^R(y)| \le C|x-y| & \text{for all } x, y \in \mathbb{R}, \\ |w^R(x) - m^R(x)| \le C, |\chi^R(x) - m^R(x)| \le C & \text{for all } x \in \mathbb{R}, \\ |w^R(x) - \chi^R(x)| \le C & \text{for all } x \in \mathbb{R}, H_0 \le \overline{A}_R \le 0. \end{cases} \tag{8.46}$$

Using Proposition 8.7, if $\overline{A} > H_0$, we know that there exists a $\gamma_0 > 0$ and a constant $C > 0$ such that for all $\gamma \in (0, \gamma_0)$, for all $x \ge r + D$, and $h \ge 0$,

$$w^R(x+h) - w^R(x) \ge (\overline{p}_+ - \gamma)h - C \quad \text{and} \quad \chi^R(x+h) - \chi^R(x) \ge (\overline{p}_+ - \gamma)h - C.$$

Similarly, for all $x \le -r - D$ and $h \ge 0$,

$$w^R(x-h) - w^R(x) \ge (-\overline{p}_- - \gamma)h - C \quad \text{and} \quad \chi^R(x-h) - \chi^R(x) \ge (-\overline{p}_- - \gamma)h - C.$$

Proceeding like before, we pass to the limit as $R \to +\infty$ in order to build a solution (w, χ) of (6.1) with $\lambda = \overline{A}$ that satisfies (6.3), (6.4) and (6.5).

Step 2: proof of iii). Let us now consider the rescaled functions $w^\varepsilon = \varepsilon w(x/\varepsilon)$ and $\chi^\varepsilon(x) = \varepsilon\chi(x/\varepsilon)$. Using (6.3), we have that

$$w^\varepsilon(x) = \varepsilon m\left(\frac{x}{\varepsilon}\right) + O(\varepsilon) \quad \text{and} \quad \chi^\varepsilon(x) = \varepsilon m\left(\frac{x}{\varepsilon}\right) + O(\varepsilon). \tag{8.47}$$

Therefore, there exists a subsequence $\varepsilon_n \to 0$ as $n \to +\infty$, such that

$$w^{\varepsilon_n}, \chi^{\varepsilon_n} \to W \quad \text{locally uniformly as } n \to +\infty, \tag{8.48}$$

with $W(0) = 0$. Proceeding as in the proof of convergence (Section 7), away from the junction point, we have that W satisfies

$$\overline{H}(W_x) = \overline{A} \quad \text{for } x \ne 0.$$

This proves (6.6). Let us now prove (6.7).

For $x < 0$, we have for all $\gamma \in (0, \gamma_0)$, if $\overline{A} > H_0$,

$$W_x \leq \overline{p}_- + \gamma,$$

where we have used (6.5). Therefore, we have $W_x = \overline{p}_-$ for $x < 0$, this equality remains valid if $\overline{A} = H_0$ (indeed, if $\overline{A} = H_0$, we have $\overline{p}_+ = \overline{p}_- = p_0 = W_x$).

For $x > 0$, we have for all $\gamma \in (0, \gamma_0)$, if $\overline{A} > H_0$,

$$W_x \geq \overline{p}_+ - \gamma,$$

where we have used (6.4). Therefore, we have that $W_x = \overline{p}_+$ for $x > 0$, this result is still valid if $\overline{A} = H_0$.

Combining these results, we obtain (6.7).

□

Theorem 8.8 (Effective flux limiter)**.** *Assume (A). We define the following set of functions,*

$$\mathcal{S} = \{(v, \zeta) \text{ s.t. } \exists \text{ a Lipschitz continuous function } m \text{ (with } m(0)=0) \\ \text{and constant } C > 0 \text{ s.t. } ||v - m||_\infty, ||\zeta - m||_\infty \leq C\}.$$

Then we have

$$\overline{A} = \inf\{\lambda \in [H_0, 0] : \exists (v, \zeta) \in \mathcal{S} \text{ solution of } (6.1)\}. \tag{8.49}$$

Proof of Theorem 8.8. Up to a sub-sequence, let $\overline{A} = \lim_{R \to +\infty} \lim_{l \to +\infty} \lambda_{l,R}$. We want to prove that $\overline{A} = \inf E$, with

$$E = \{\lambda \in [H_0, 0] : \exists (v, \zeta) \in \mathcal{S} \text{ solution of } (6.1)\}.$$

We argue by contradiction and assume that there exists $\lambda \in E$ such that $\lambda < \overline{A}$. We denote by $(v^\lambda, \zeta^\lambda)$ a solution of (6.1) associated to λ. Arguing as in the proof of Theorem 6.1, Step 2, we deduce that the functions

$$v_\lambda^\varepsilon(x) = \varepsilon v^\lambda\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \zeta_\lambda^\varepsilon(x) = \varepsilon \zeta^\lambda\left(\frac{x}{\varepsilon}\right) \tag{8.50}$$

have a limit W^λ (with $W^\lambda(0) = 0$) which satisfies

$$\overline{H}(W_x^\lambda) = \lambda \quad \text{for } x \neq 0.$$

This means that for all $x > 0$, we have

$$W_x^\lambda \leq p_+^\lambda < \overline{p}_+ \quad \text{with } \overline{H}(p_+^\lambda) = \overline{H}^+(p_+^\lambda) = \lambda. \tag{8.51}$$

Similarly, for all $x < 0$, we have

$$W_x^\lambda \geq p_-^\lambda > \overline{p}_- \quad \text{with } \overline{H}(p_-^\lambda) = \overline{H}^-(p_-^\lambda) = \lambda. \tag{8.52}$$

These inequalities imply that for all $\gamma > 0$, there exists a constant $\tilde{C}_\gamma$ such that

$$v^\lambda(x), \zeta^\lambda(x) \leq \begin{cases} (p_+^\lambda + \gamma)x + \tilde{C}_\gamma & \text{for } x > 0, \\ (p_-^\lambda - \gamma)x + \tilde{C}_\gamma & \text{for } x < 0. \end{cases} \tag{8.53}$$

Using Theorem 6.1 (ii), we have for γ small enough,

$$v^\lambda \leq w \quad \text{and} \quad \zeta^\lambda \leq \chi \quad \text{for } |x| \geq \tilde{R}.$$

This implies that there exists a constant $C_{\tilde{R}}$ such that for all $x \in \mathbb{R}$, we have

$$v^\lambda(x) < w(x) + C_{\tilde{R}} \quad \text{and} \quad \zeta^\lambda(x) < \chi(x) + C_{\tilde{R}}.$$

Let us now introduce two functions (u, ξ) and (u^λ, ξ^λ), defined by

$$\begin{cases} u(t,x) = w(x) + C_{\tilde{R}} - \overline{A}t, \\ \xi(t,x) = \chi(x) + C_{\tilde{R}} - \overline{A}t, \end{cases} \quad \text{and} \quad \begin{cases} u^\lambda(t,x) = v^\lambda(x) - \lambda t, \\ \xi^\lambda(t,x) = \zeta^\lambda(x) - \lambda t. \end{cases}$$

Both functions are solutions of (3.3) (with $\varepsilon = 1$) and

$$u^\lambda(0,x) \le u(0,x) \quad \text{and} \quad \xi^\lambda(0,x) \le \xi(0,x).$$

Using the comparison principle (Proposition 4.7), we obtain

$$v^\lambda(x) - \lambda t \le w(x) - \overline{A}t + C_{\tilde{R}}.$$

Passing to the limit as t goes to infinity, we get $\overline{A} \le \lambda$, which is a contradiction.

□

9 Link between the system of ODEs and the PDE

This section is devoted to the proof of Theorem 3.3, which is a direct application of our convergence result, Theorem 3.2 joint to the following result.

Theorem 9.1. *For $\varepsilon = 1$, (ρ, σ) defined by (2.2) and* (3.2) *is a discontinuous viscosity solution of the following equation*

$$\begin{cases} \rho_t + M\left(\rho(t,x), [\sigma(t,\cdot)]\right)(x) \cdot |\rho_x| = 0 \\ \sigma_t + L\left(x, \sigma(t,x), [\rho(t,\cdot)]\right)(x) \cdot |\sigma_x| = 0 \end{cases} \quad \text{for } (t,x) \in (0,+\infty) \times \mathbb{R}. \tag{9.1}$$

The proof of Theorem 9.1 is given in Appendix B. Let us use Theorem 9.1 to do the proof of Theorem 3.3.

Proof of Theorem 3.3. We define two functions u_0 and ξ_0^ε satisfying (A0) such that

$$\rho^\varepsilon(0,x) = \rho_0^\varepsilon(x) = \varepsilon \left\lfloor \frac{u_0(x)}{\varepsilon} \right\rfloor \quad \text{and} \quad \sigma^\varepsilon(0,x) = \sigma_0^\varepsilon(x) = \varepsilon \left\lfloor \frac{\xi_0^\varepsilon(x)}{\varepsilon} \right\rfloor.$$

By construction we have

$$(\rho_0^\varepsilon)^*(x) = \rho_0^\varepsilon(x) \le u_0(x) < (\rho_0^\varepsilon)_*(x) + \varepsilon,$$
$$(\sigma_0^\varepsilon)^*(x) = \sigma_0^\varepsilon(x) \le \xi_0^\varepsilon(x) < (\sigma_0^\varepsilon)_*(x) + \varepsilon.$$

Using the fact that $(\rho^\varepsilon, \sigma^\varepsilon)$ is a viscosity solution of (3.3) and the comparison principle (Proposition 4.7) we deduce that (with $(u^\varepsilon, \xi^\varepsilon)$ the continuous solution of (3.3))

$$\rho^\varepsilon(t,x) \le u^\varepsilon(t,x) \le (\rho^\varepsilon)_*(t,x) + \varepsilon \quad \text{and} \quad \sigma^\varepsilon(t,x) \le \xi^\varepsilon(t,x) \le (\sigma^\varepsilon)_*(t,x) + \varepsilon.$$

and therefore

$$u^\varepsilon(t,x) - \varepsilon \le \rho^\varepsilon(t,x) \le u^\varepsilon(t,x) \quad \text{and} \quad \xi^\varepsilon(t,x) - \varepsilon \le \sigma^\varepsilon(t,x) \le \xi^\varepsilon(t,x).$$

Passing to the limit as $\varepsilon \to 0$, we get that $\rho^\varepsilon, \sigma^\varepsilon \to u^0$, which ends the proof of Theorem 3.3.

□

Appendix A Analysis of system (3.1)

In this section we present some properties of the solution $(U_i, \Xi_i)_{i\in\mathbb{Z}}$ of

$$\begin{cases} \dot{U}_j(t) = \alpha\left(\Xi_j(t) - U_j(t)\right) \\ \dot{\Xi}_j(t) = \alpha\left(U_j(t) - \Xi_j(t)\right) + 2V\left(U_{j+1}(t) - U_j(t)\right) \cdot \phi\left(U_j(t)\right). \end{cases} \tag{A.1}$$

We couple system (A.1) with an initial condition $(U_i(0), \Xi_i(0))_i$ that satisfy the following assumption.

(A0') (Initial conditions for (A.1)**).** For all $i \in \mathbb{Z}$,

$$0 \le \Xi_i(0) - U_i(0) \le \frac{V_{max}}{\alpha}, \quad U_{i+1}(0) - \Xi_i(0) \ge h_0, \quad \text{and} \quad U_{i+1}(0) - U_i(0) \le h_{max}. \tag{A.2}$$

Proposition A.1 (Bounds on the velocities of the vehicles)**.** *Assume (A) and (A0'), then the solution* $(U_i, \Xi_i)_i$ *of* (A.1) *satisfies for all* $i \in \mathbb{Z}$

$$0 \le \Xi_i(t) - U_i(t) \le \frac{V_{max}}{\alpha} \quad \text{for all } t > 0. \tag{A.3}$$

Proof. Let us consider the equation satisfied by $\Xi_i - U_i$,

$$\begin{cases} \dfrac{d(\Xi_i - U_i)}{dt} = -2\alpha\left(\Xi_i - U_i\right) + 2V\left(U_{j+1} - U_j\right) \cdot \phi\left(U_j\right) \quad \text{for all } t > 0, \\ 0 \le \Xi_i(0) - U_i(0) \le \dfrac{V_{max}}{\alpha}. \end{cases}$$

Step 1: proof of the upper bound in (A.3)**.** Using assumptions (A1), (A4), and (A6), we notice that $\Xi_i - U_i$ is a sub-solution of

$$\begin{cases} \dot{z} = -2\alpha z + 2V_{max}, \\ z(0) = \dfrac{V_{max}}{\alpha}. \end{cases} \tag{A.4}$$

By comparison, we have

$$\Xi_i(t) - U_i(t) \le z(t) = \frac{V_{max}}{\alpha} \quad \text{for all } t \ge 0.$$

Step 2: proof of the lower bound in (A.3)**.** Using assumptions (A1), (A3), and (A6), we notice that $\Xi_i - U_i$ is a super-solution of

$$\begin{cases} \dot{z} = -2\alpha z, \\ z(0) = 0. \end{cases} \tag{A.5}$$

By comparison, we have

$$\Xi_i(t) - U_i(t) \ge z(t) = 0 \quad \text{for all } t \ge 0.$$

This ends the proof of Proposition A.1.

□

Proposition A.2 (Conservation of the order in (A.1))**.** *Assume (A) and (A0'), then the solution* $(U_i, \Xi_i)_i$ *of* (A.1) *satisfies for all* $i \in \mathbb{Z}$,

$$U_{i+1}(t) - \Xi_i(t) \ge h_0 \quad \text{for all } t > 0. \tag{A.6}$$

In particular, using Proposition A.1, this result implies that

$$U_{i+1}(t) - U_i(t) \ge h_0 \quad \text{and} \quad \Xi_{i+1}(t) - \Xi_i(t) \ge h_0 \quad \text{for all } t > 0. \tag{A.7}$$

Proof. We will prove that for all $\delta > 0$ small, we have

$$U_{i+1}(t) - \Xi_i(t) \geq h_0 - \delta \quad \text{for all } t > 0.$$

Then passing to the limit as δ goes to 0 we will obtain (A.6).

Let $\delta > 0$, we argue by contradiction and assume there exists a time

$$t^* = \inf\{t, \text{ s.t. } \exists j \in \mathbb{Z} \text{ s.t. } U_{j+1}(t) - \Xi_j(t) = h_0 - \delta\}.$$

Let us consider $j \in \mathbb{Z}$ such that $U_{j+1}(t^*) - \Xi_j(t^*) = h_0 - \delta$. By continuity, there exists a time $t_0 \in [0, t^*)$ such that

$$U_{j+1}(t_0) - \Xi_j(t_0) = h_0 \quad \text{and} \quad U_{j+1}(t) - \Xi_j(t) \in [h_0 - \delta, h_0] \quad \text{for all } t \in [t_0, t^*].$$

Using Proposition A.1, in particular that $U_j \leq \Xi_j$, and assumption (A7) combined with Remark 2.2, we have that

$$\alpha(U_j - \Xi_j) + 2V(U_{j+1} - U_j) \cdot \phi(U_j) \leq 2V(U_{j+1} - \Xi_j) \cdot \phi(\Xi_j) \leq 2V(h_0) \cdot \phi(\Xi_j) = 0. \quad \text{(A.8)}$$

This implies that (U_j, Ξ_j) satisfies for all $t \in [t_0, t^*]$,

$$\begin{cases} \dot{U}_j = \alpha(\Xi_j - U_j) \\ \dot{\Xi}_j \leq 0, \end{cases} \quad \text{with} \quad \begin{cases} U_j(t_0) \leq \Xi_j(t_0) \\ \Xi_j(t_0) = U_{j+1}(t_0) - h_0. \end{cases}$$

Therefore, we have for all $t \in [t_0, t^*]$

$$\Xi_j(t) \leq U_{j+1}(t_0) - h_0.$$

Using again Proposition A.1, in particular that the functions $(U_i)_i$ are non-decreasing in time, we obtain that

$$\Xi_j(t^*) \leq U_{j+1}(t^*) - h_0,$$

which is a contradiction. This ends the proof of Proposition A.2.

□

Proposition A.3 (Maximal distance between two vehicles)**.** *Assume (A) and (A0'), then the solution* $(U_i, \Xi_i)_i$ *of* (A.1) *satisfies for all* $i \in \mathbb{Z}$*,*

$$U_{i+1}(t) - U_i(t) \leq h_{max} + \frac{3V_{max}}{2\alpha} + \frac{2r}{\phi_0} \quad \text{for all } t > 0. \quad \text{(A.9)}$$

In particular, using Proposition A.1, we have that for all $i \in \mathbb{Z}$*,*

$$U_{i+1}(t) - \Xi_i(t) \leq h_{max} + \frac{3V_{max}}{2\alpha} + \frac{2r}{\phi_0} \quad \text{for all } t > 0. \quad \text{(A.10)}$$

Proof. We will prove that for all $\delta > 0$ small, we have for all $i \in \mathbb{Z}$,

$$U_{i+1}(t) - U_i(t) \leq h_{max} + \frac{3V_{max}}{2\alpha} + \frac{2r}{\phi_0} + \delta \quad \text{for all } t > 0. \quad \text{(A.11)}$$

Passing to the limit in the previous inequality as δ goes to 0, we will obtain (A.9).

Let $\delta > 0$, we argue by contradiction and assume there exists a time

$$t^* = \inf\left\{t \text{ s.t. } \exists j \in \mathbb{Z} \text{ s.t. } U_{j+1}(t) - U_j(t) > h_{max} + \frac{2r}{\phi_0} + \frac{3V_{max}}{2\alpha} + \delta\right\}.$$

Let us consider $j \in \mathbb{Z}$ such that $U_{j+1}(t^*) - U_j(t^*) = h_{max} + \frac{2r}{\phi_0} + \frac{3V_{max}}{2\alpha} + \delta$. By continuity and (A0'), there exists a time $t_0 \in [0, t^*)$ such that

$$U_{j+1}(t_0) - U_j(t_0) = h_{max} \tag{A.12}$$

and

$$U_{j+1}(t) - U_j(t) \in \left[h_{max}, h_{max} + \frac{2r}{\phi_0} + \frac{3V_{max}}{2\alpha} + \delta\right] \quad \text{for all } t \in [t_0, t^*].$$

We distinguish three cases.

Case 1: $U_j(t_0) \in [-r, r]$**.** The couple (U_j, Ξ_j) satisfy for all $t \in [t_0, t^*]$

$$\begin{cases} \dot{U}_j = \alpha(\Xi_j - U_j) \\ \dot{\Xi}_j = \alpha(U_j - \Xi_j) + 2V_{max} \cdot \phi(U_j), \end{cases} \quad \text{with} \quad \begin{cases} U_j(t_0) = U_{j+1}(t_0) - h_{max} \\ 0 \le \Xi_j(t_0) - U_j(t_0) \le \frac{V_{max}}{\alpha}. \end{cases} \tag{A.13}$$

In order to compare the distance $U_{j+1} - U_j$ when U_j is inside the perturbation, we consider the worst case scenario where the vehicle j advances at a speed $V_{max}\phi_0$ and $j+1$ advances at a speed V_{max}, until $U_j \ge r$ (meaning that the vehicle j is outside the perturbation). To be more exact, we notice that the couple (U_j, Ξ_j) is a super-solution of the following system

$$\begin{cases} \dot{v} = \alpha(\zeta - v) \\ \dot{\zeta} = \alpha(v - \zeta) + 2V_{max}\phi_0, \end{cases} \quad \text{with} \quad \begin{cases} v(t_0) = U_{j+1}(t_0) - h_{max} \\ \zeta(t_0) = v(t_0). \end{cases} \tag{A.14}$$

Computing the solution of (A.14) we get

$$\begin{cases} v(t) = \frac{V_{max}\phi_0}{2\alpha} e^{-2\alpha(t-t_0)} - \frac{V_{max}\phi_0}{2\alpha} + V_{max}\phi_0(t - t_0) + v(t_0) \\ \zeta(t) = -\frac{V_{max}\phi_0}{2\alpha} e^{-2\alpha(t-t_0)} + \frac{V_{max}\phi_0}{2\alpha} + V_{max}\phi_0(t - t_0) + v(t_0) \end{cases} \tag{A.15}$$

By comparison, we obtain that

$$U_j(t) \ge v(t) = \frac{V_{max}\phi_0}{2\alpha} e^{-2\alpha(t-t_0)} - \frac{V_{max}\phi_0}{2\alpha} + V_{max}\phi_0(t - t_0) + v(t_0). \tag{A.16}$$

Let $\hat{t} = \frac{1}{V_{\max}\phi_0}\left(\frac{V_{max}\phi_0}{2\alpha} + r - U_j(t_0)\right) + t_0$. Using (A.16), we have that $U_j(\hat{t}) \ge r$. We now prove that $\hat{t} < t^*$. In fact, for all $t \in [t_0, \hat{t}]$, we have

$$\begin{aligned} U_{j+1}(t) - U_j(t) &\le U_{j+1}(\hat{t}) - U_j(t_0) \le V_{max}(\hat{t} - t_0) + U_{j+1}(t_0) - U_j(t_0) \\ &= V_{max}\left(\frac{1}{V_{\max}\phi_0}\left(\frac{V_{max}\phi_0}{2\alpha} + r - U_j(t_0)\right)\right) + U_{j+1}(t_0) - U_j(t_0) \\ &\le \frac{V_{max}}{2\alpha} + \left(\frac{r - U_j(t_0)}{\phi_0}\right) + h_{max} \\ &\le \frac{V_{max}}{2\alpha} + \frac{2r}{\phi_0} + h_{max}, \end{aligned}$$

where we have used Proposition A.1 for the first line. From the previous inequality and the definition of t^*, we deduce that $\hat{t} < t^*$.

The couple (U_j, Ξ_j) satisfies for all $t \in [\hat{t}, t^*]$,

$$\begin{cases} \dot{U}_j = \alpha(\Xi_j - U_j) \\ \dot{\Xi}_j = \alpha(U_j - \Xi_j) + 2V_{max}, \end{cases} \tag{A.17}$$

with

$$\begin{cases} h_{max} \leq U_{j+1}(\hat{t}) - U_j(\hat{t}) \leq h_{max} + \dfrac{2r}{\phi_0} + \dfrac{V_{max}}{2\alpha} \\ 0 \leq \Xi_j(\hat{t}) - U_j(\hat{t}) \leq \dfrac{V_{max}}{\alpha}. \end{cases} \tag{A.18}$$

We can easily compute the explicit form of the solution of (A.18),

$$U_j(t) = \Big(\frac{V_{max}}{\alpha} - \Xi_j(\hat{t}) + U_j(\hat{t})\Big)\frac{e^{-2\alpha(t-\hat{t})}}{2} - \frac{V_{max}}{2\alpha} + V_{max}(t-\hat{t}) + \frac{1}{2}\left(\Xi_j(\hat{t}) + U_j(\hat{t})\right)$$

and

$$\Xi_j(t) = \Big(\Xi_j(\hat{t}) - U_j(\hat{t}) - \frac{V_{max}}{\alpha}\Big)\frac{e^{-2\alpha(t-\hat{t})}}{2} + \frac{V_{max}}{2\alpha} + V_{max}(t-\hat{t}) + \frac{1}{2}\left(\Xi_j(\hat{t}) + U_j(\hat{t})\right).$$

Using Proposition A.1, for all $t \in [\hat{t}, t^*]$, we have that

$$U_{j+1}(t) \leq V_{max}(t-\hat{t}) + U_{j+1}(\hat{t}). \tag{A.19}$$

Therefore, combining the previous results, we have for all $t \in [\hat{t}, t^*]$

$$\begin{aligned} U_{j+1}(t) - U_j(t) \leq & V_{max}(t-\hat{t}) + U_{j+1}(\hat{t}) - V_{max}(t-\hat{t}) - \frac{1}{2}\left(\Xi_j(\hat{t}) + U_j(\hat{t})\right) \\ & - \Big(\frac{V_{max}}{\alpha} - \Xi_j(\hat{t}) + U_j(\hat{t})\Big)\frac{e^{-2\alpha(t-\hat{t})}}{2} + \frac{V_{max}}{2\alpha} \\ \leq & U_{j+1}(\hat{t}) - \frac{1}{2}\left(\Xi_j(\hat{t}) + U_j(\hat{t})\right) + \frac{V_{max}}{2\alpha} \\ \leq & U_{j+1}(\hat{t}) - U_j(\hat{t}) + \frac{V_{max}}{2\alpha} \\ \leq & h_{max} + \frac{2r}{\phi_0} + \frac{V_{max}}{\alpha}, \end{aligned}$$

where we have used Proposition A.1 for the second and third inequality and we have used (A.18) for the last inequality. The previous inequality remains valid for $t = t^*$ which gives us a contradiction.

Case 2: $U_j(t_0) > r$**.** In this case, the couple (U_j, Ξ_j) satisfies system (A.17) for all $t \in (t_0, t^*]$, with the following initial conditions

$$\begin{cases} U_j(t_0) = U_{j+1}(t_0) - h_{max} \\ 0 \leq \Xi_j(t_0) - U_j(t_0) \leq \dfrac{V_{max}}{\alpha}. \end{cases} \tag{A.20}$$

As above, the explicit solution of (A.17)-(A.20) has the following form,

$$U_j(t) = \Big(\frac{V_{max}}{\alpha} - \Xi_j(t_0) + U_j(t_0)\Big)\frac{e^{-2\alpha(t-t_0)}}{2} - \frac{V_{max}}{2\alpha} + V_{max}(t-t_0) + \frac{1}{2}\left(\Xi_j(t_0) + U_j(t_0)\right)$$

and

$$\Xi_j(t) = \Big(\Xi_j(t_0) - U_j(t_0) - \frac{V_{max}}{\alpha}\Big)\frac{e^{-2\alpha(t-t_0)}}{2} + \frac{V_{max}}{2\alpha} + V_{max}(t-t_0) + \frac{1}{2}\left(\Xi_j(t_0) + U_j(t_0)\right).$$

Arguing as above, we will obtain $U_{j+1}(t^*) - U_j(t^*) \leq h_{max} + \dfrac{V_{max}}{2\alpha}$ which is a contradiction.

Case 3: $U_j(t_0) < -r$**.** We treat this case in 3 steps.

Step 1: left of the perturbation. We denote by

$$\hat{t} = \inf \{t \geq t_0 \text{ s.t. } U_j(t) = -r\}.$$

For all $t \in [t_0, \hat{t}]$, the couple (U_j, Ξ_j) satisfies (A.17)-(A.20) and therefore has the same form as the one presented in Case 2. In particular, for all $t \in [t_0, \hat{t}]$, we have

$$U_{j+1}(t) - U_j(t) \leq h_{max} + \frac{V_{max}}{2\alpha}. \tag{A.21}$$

This implies that $\hat{t} < t^*$.

Step 2: inside the perturbation. In the interval $[\hat{t}, t^*]$, the couple (U_j, Ξ_j) satisfies (A.13) with the following initial condition

$$\begin{cases} U_{j+1}(\hat{t}) - U_j(\hat{t}) \leq h_{max} + \dfrac{V_{max}}{2\alpha} \\ 0 \leq \Xi_j(\hat{t}) - U_j(\hat{t}) \leq \dfrac{V_{max}}{\alpha}. \end{cases}$$

The couple (U_j, Ξ_j) is a super-solution of

$$\begin{cases} \dot{v} = \alpha(\zeta - v) \\ \dot{\zeta} = \alpha(v - \zeta) + 2V_{max}\phi_0, \end{cases} \quad \text{with} \quad \begin{cases} v(\hat{t}) = U_{j+1}(\hat{t}) + h_{max} + \dfrac{V_{max}}{2\alpha} \\ \zeta(\hat{t}) = v(\hat{t}). \end{cases} \tag{A.22}$$

Computing the solution of (A.22), and by comparison, for all $t \in [\hat{t}, t^*]$, we have

$$U_j(t) \geq \frac{V_{max}\phi_0}{2\alpha} e^{-2\alpha(t-\hat{t})} - \frac{V_{max}\phi_0}{2\alpha} + V_{max}\phi_0(t - \hat{t}) + v(\hat{t}).$$

Let $\tilde{t} = \dfrac{1}{V_{\max}\phi_0}\left(\dfrac{V_{max}\phi_0}{2\alpha} + r - U_j(\hat{t})\right) + \hat{t}$. Using (A.16), we have that $U_j(\tilde{t}) \geq r$. We now prove that $\tilde{t} < t^*$. We recall that $U_j(\hat{t}) = -r$. In fact, for all $t \in [\hat{t}, \tilde{t}]$, we have

$$\begin{aligned} U_{j+1}(t) - U_j(t) &\leq U_{j+1}(\tilde{t}) - U_j(\hat{t}) \leq V_{max}(\tilde{t} - \hat{t}) + U_{j+1}(\hat{t}) - U_j(\hat{t}) \\ &= V_{max}\left(\frac{1}{V_{\max}\phi_0}\left(\frac{V_{max}\phi_0}{2\alpha} + r - U_j(\hat{t})\right)\right) + U_{j+1}(\hat{t}) - U_j(\hat{t}) \\ &\leq \frac{V_{max}}{\alpha} + \frac{2r}{\phi_0} + h_{max}, \end{aligned}$$

where we have used Proposition A.1 for the first line. From the previous inequality and the definition of t^*, we deduce that $\tilde{t} < t^*$.

Step 3: right of the perturbation. In the interval $[\tilde{t}, t^*]$, the couple (U_j, Ξ_j) satisfies (A.17), with the following initial condition

$$U_{j+1}(\tilde{t}) - U_j(\tilde{t}) \leq \frac{V_{max}}{\alpha} + \frac{2r}{\phi_0} + h_{max}.$$

Proceeding like before, we can prove that for all $t \in [\tilde{t}, t^*]$, we have

$$U_{j+1}(t) - U_j(t) \leq \frac{3V_{max}}{2\alpha} + \frac{2r}{\phi_0} + h_{max},$$

which gives us a contradiction for $t = t^*$. This ends the proof of Proposition A.3.

□

Appendix B Proof of Theorem 9.1

Before we give the proof of Theorem 9.1, we need the following result.

Lemma B.1 (Link between the velocities)**.** *Assume (A). Let* $((U_j)_j, (\Xi_j)_j)$ *be the solution of (3.1) with an initial condition* $(U_j(0), \Xi_j(0))_j$ *satisfying (A0'). Then we have*

$$\dot{U}_j(t) = -M\left(u\left(t, U_j(t)\right), \left[\xi\left(t, \cdot\right)\right]\right)\left(U_j(t)\right) \tag{B.1}$$

and

$$\dot{\Xi}_j(t) = -L\left(\Xi_j(t), \xi\left(t, \Xi_j(t)\right), \left[u(t, \cdot)\right]\right)\left(\Xi_j(t)\right), \tag{B.2}$$

where u *and* ξ *are continuous functions such that*

$$\begin{cases} u(t,x) = \rho_*(t,x) = \rho(t,x) \text{ for } x = U_j(t),\ j \in \mathbb{Z}, \\ u \text{ is decreasing in } x, \end{cases} \tag{B.3}$$

$$\begin{cases} \xi(t,x) = \sigma_*(t,x) = \sigma(t,x) \text{ for } x = \Xi_j(t),\ j \in \mathbb{Z}, \\ \xi \text{ is decreasing in } x, \end{cases} \tag{B.4}$$

where ρ *and* σ *are defined respectively in* (2.2) *and* (3.2) *(with* $\varepsilon = 1$*).*

Proof. We drop the time dependence to simplify the presentation. Let $j \in \mathbb{Z}$. We recall that we chose $D = h_{max} + 3V_{max}/(2\alpha) + 2r/\phi_0$. Using the fact that $u(t, U_j(t)) = -(j+1)$ and (B.3), we have for all $z \in [0, +\infty)$,

$$\begin{cases} \xi(U_j + z) - u(U_j) > \xi(\Xi_j) - u(U_j) = 0 & \text{if } z \in [0, \Xi_j - U_j) \\ \xi(U_j + z) - u(U_j) \le 0 & \text{if } z \in [\Xi_j - U_j, +\infty). \end{cases}$$

Using Proposition A.1, in particular that $\Xi_j - U_j \le D$, we have

$$\begin{aligned} &M\left(u\left(t, U_j(t)\right), \left[\xi\left(t, \cdot\right)\right]\right)\left(U_j(t)\right) = \int_0^D E\left(\xi\left(U_j + z\right) - u\left(U_j\right)\right) dz \\ &= \int_0^{\Xi_j - U_j} E\left(\xi\left(U_j + z\right) - u\left(U_j\right)\right) dz + \int_{\Xi_j - U_j}^D E\left(\xi\left(U_j + z\right) - u\left(U_j\right)\right) dz \\ &= -\alpha\left(\Xi_j - U_j\right). \end{aligned}$$

Combining this result with (3.1), we obtain (B.1). We now turn to the proof of (B.2).

We will begin by computing $K\left(\xi\left(t, \Xi_j(t)\right), \left[u(t, \cdot)\right]\right)\left(\Xi_j(t)\right)$. Using the fact that $\xi\left(t, \Xi(t)\right) = -(j+1)$ and (B.4), we have for all $z \in [0, +\infty)$,

$$\begin{cases} u(\Xi_j - z) - \xi(\Xi_j) < u(U_j) - \xi(\Xi_j) = 0 & \text{if } z \in [0, \Xi_j - U_j) \\ u(\Xi_j - z) - \xi(\Xi_j) \ge 0 & \text{if } z \in [\Xi_j - U_j, +\infty). \end{cases}$$

Thanks to Proposition A.1, this implies that

$$K\left(\xi\left(t, \Xi_j(t)\right), \left[u\left(t, \cdot\right)\right]\right)\left(\Xi_j(t)\right) = \int_0^{\Xi_j - U_j} F\left(u\left(\Xi_j - z\right) - \xi\left(\Xi_j\right)\right) dz = \Xi_j - U_j.$$

We now turn to the computation of $N\left(\xi\left(t, \Xi_j(t)\right), \left[u\left(t, \cdot\right)\right]\right)\left(\Xi_j(t)\right)$. We recall that thanks to Proposition A.2, we have $U_{j+1} - \Xi_j \ge h_0$. In particular, we have that

$$\begin{cases} u(\Xi_j + z) - \xi(\Xi_j) > u(U_{j+1}) - \xi(\Xi_j) = -1 & \text{if } z \in [0, U_{j+1} - \Xi_j) \\ u(\Xi_j + z) - \xi(\Xi_j) \le -1 & \text{if } z \in [U_{j+1} - \Xi_j, +\infty). \end{cases}$$

Once more thanks to Proposition A.3, we obtain

$$N\left(\xi\left(t, \Xi_j(t)\right), \left[u\left(t, \cdot\right)\right]\right)\left(\Xi_j(t)\right) = \int_0^{U_{j+1} - \Xi_j} I\left(u\left(\Xi_j + z\right) - \xi\left(\Xi_j\right)\right) dz = U_{j+1} - \Xi_j.$$

Combining the previous results with (3.14) and (3.1), we obtain (B.2).

□

Proof of Theorem 9.1. We remark that thanks to (B.3) and (B.4), we have for $x = U_j(t)$ and $y = \Xi_j(t),\ \ j \in \mathbb{Z}$,

$$\tilde{M}\left(\rho_*(t,x), [\sigma_*(t,\cdot)]\right)(x) = \tilde{M}\left(u(t,x), [\xi(t,\cdot)]\right)(x) \geq M\left(u(t,x), [\xi(t,\cdot)]\right)(x),$$

and

$$\tilde{L}\left(y, \sigma_*(t,y), [\rho_*(t,\cdot)]\right)(y) = \tilde{L}\left(y, \xi_*(t,y), [u(t,\cdot)]\right)(y) \geq L\left(y, \xi_*(t,y), [u(t,\cdot)]\right)(y).$$

Using Lemma B.1, and Definition 4.1, we can see that (ρ_*, σ_*) is a discontinuous viscosity supersolution of (9.1). We obtain a similar result for (ρ^*, σ^*), therefore, (ρ, σ) is a discontinuous viscosity solution of (9.1).

□

ACKNOWLEDGMENTS

This project was co-financed by the European Union with the European regional development fund (ERDF) and by the Haute-Normandie Regional Council via the M2NUM project and by ANR HJNet (ANR-12-BS01-0008-01).

References

[1] Y. Achdou and N. Tchou, *Hamilton-jacobi equations on networks as limits of singularly perturbed problems in optimal control: dimension reduction*, Communications in Partial Differential Equations, 40 (2015), pp. 652–693.

[2] O. Alvarez and A. Tourin, *Viscosity solutions of nonlinear integro-differential equations*, in Annales de l'Institut Henri Poincaré. Analyse non linéaire, vol. 13, Elsevier, 1996, pp. 293–317.

[3] A. Aw, A. Klar, M. Rascle, and T. Materne, *Derivation of continuum traffic flow models from microscopic follow-the-leader models*, SIAM Journal on Applied Mathematics, 63 (2002), pp. 259–278.

[4] M. Bando, K. Hasebe, A. Nakayama, A. Shibata, and Y. Sugiyama, *Dynamical model of traffic congestion and numerical simulation*, Physical Review E, 51 (1995), p. 1035.

[5] G. Barles, *Solutions de viscosité des équations de Hamilton-Jacobi*, Springer Verlag, 1994.

[6] M. G. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bulletin of the American Mathematical Society, 27 (1992), pp. 1–67.

[7] F. Da Lio, N. Forcadel, and R. Monneau, *Convergence of a non-local eikonal equation to anisotropic mean curvature motion. Application to dislocation dynamics*, J. Eur. Math. Soc. (JEMS), 10 (2008), pp. 1061–1104.

[8] M. Di Francesco and M. D. Rosini, *Rigorous derivation of nonlinear scalar conservation laws from follow-the-leader type models via many particle limit*, Arch. Ration. Mech. Anal., 217 (2015), pp. 831–871.

[9] N. Forcadel, C. Imbert, and R. Monneau, *Homogenization of fully overdamped frenkel-kontorova models*, Journal of Differential Equations, 246 (2009), pp. 1057–1097.

[10] ——, *Homogenization of some particle systems with two-body interactions and of the dislocation dynamics*, Discrete Contin. Dyn. Syst., 23 (2009), pp. 785–826.

[11] ——, *Homogenization of accelerated frenkel-kontorova models with n types of particles*, Transactions of the American Mathematical Society, 364 (2012), pp. 6187–6227.

[12] N. Forcadel and W. Salazar, *Homogenization of second order discrete model and application to traffic flow*, Differential and Integral Equations, 28 (2015), pp. 1039–1068.

[13] N. Forcadel and W. Salazar, *A junction condition by specified homogenization of a discrete model with a local perturbation and application to traffic flow*, <hal-01097085>, (2015).

[14] G. Galise, C. Imbert, and R. Monneau, *A junction condition by specified homogenization and application to traffic lights*, Anal. PDE, 8 (2015), pp. 1891–1929.

[15] J. M. Greenberg, *Extensions and amplifications of a traffic model of Aw and Rascale*, SIAM J. Appl. Math, 62 (2001), pp. 729–745.

[16] D. Helbing, *From microscopic to macroscopic traffic models*, in A perspective look at nonlinear media, Springer, 1998, pp. 122–139.

[17] M. Herty and L. Pareschi, *Fokker-Planck asymptotics for traffic flow models*, Kinet. Relat. Models, 3 (2010), pp. 165–179.

[18] C. Imbert, *A non-local regularization of first order hamilton–jacobi equations*, Journal of Differential Equations, 211 (2005), pp. 218–246.

[19] C. Imbert and R. Monneau, *Flux-limited solutions for quasi-convex hamilton-jacobi equations on networks*, <hal-00832545>, (2014).

[20] C. Imbert, R. Monneau, and E. Rouy, *Homogenization of first order equations with (u/ε)-periodic hamiltonians part ii: Application to dislocations dynamics*, Communications in Partial Differential Equations, 33 (2008), pp. 479–516.

[21] H. Ishii and S. Koike, *Viscosity solutions for monotone systems of second–order elliptic pdes*, Communications in partial differential equations, 16 (1991), pp. 1095–1128.

[22] W. Knödel, *Graphentheoretische Methoden und ihre Anwendungen*, Econometrics and Operations Research, XIII, Springer-Verlag, Berlin-New York, 1969.

[23] H. Lee, H.-W. Lee, and D. Kim, *Macroscopic traffic models from microscopic car-following models*, Physical Review E, 64 (2001), p. 056126.

[24] M. J. Lighthill and G. B. Whitham, *On kinematic waves. ii. a theory of traffic flow on long crowded roads*, Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences, 229 (1955), pp. 317–345.

[25] P. L. Lions, *Lectures at collège de france*, 2013-2014.

[26] P. I. Richards, *Shock waves on the highway*, Operations research, 4 (1956), pp. 42–51.