Homogenization of second order discrete model and application to traffic flow

N. Forcadel[1], W. Salazar[1]

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Abstract

The goal of this paper is to derive a traffic flow macroscopic models from microscopic models. At the microscopic scales, we consider a Bando model, of the type following the leader, i.e. the acceleration of each vehicle depends on the distance to the vehicle in front of it. We take into account the possibility that each driver can have different characteristics such as sensibility to other drivers or optimal velocities. After rescaling, we prove that the solution of this system of ODEs converges to the solution of a macroscopic homogenized Hamilton-Jacobi equation which can be seen as a LWR (Lighthill-Whitham-Richards) model.

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1 Introduction

The goal of this paper is to obtain an homogenization result for a traffic flow model. More precisely, we are interested in a discrete model (of type "following the leader") which describes the dynamics of vehicles on a straight road. The microscopic model we consider was introduced by Bando *et al* [1] and is an optimal velocity model. The goal is then to describe the collective behaviour of the vehicles (in term of the density of vehicles) as the number of vehicles per unit length goes to infinity. We will see in particular that this problem can be seen as an homogenization result. Let us mention that the theory of homogenization for periodic Hamilton-Jacobi equations has known an important development since the pioneer works of Lions, Papanicolaou, Varadhan [16] and Evans [6]. We would like in particular to mention [4] which is concerned with the homogenization of system and [7, 8, 9, 10] for the homogenization of non-local equations (or systems).

1.1 General model with n_0 types of drivers

We begin by recalling the model introduced in [1]. We consider that we have $n_0 \in \mathbb{N}$ types of drivers (or vehicles), and we consider the following optimal velocity model, for all $j \in \mathbb{Z}$ and $t \geq 0$,

$$\ddot{U}_j(t) = a_j(V_j(U_{j+1}(t) - U_j(t)) - \dot{U}_j), \tag{1.1}$$

where U_j denotes the position of j-th vehicle, $\dot{U}_j$ is its velocity and $\ddot{U}_j$ its acceleration. The coefficients a_j are the sensibilities of the drivers and V_j are called optimal velocity functions (OVF) and depends on the driver.

[1]INSA de Rouen, Normandie Université, Labo. de Mathématiques de l'INSA - LMI (EA 3226 - FR CNRS 3335) 685 Avenue de l'Université, 76801 St Etienne du Rouvray cedex. France

To simplify the study and in order to be able to get homogenisation, we impose the following periodic conditions

$$a_{j+n_0} = a_j \quad \text{and} \quad V_{j+n_0} = V_j \quad \text{for all } j \in \mathbb{Z}.$$

The model we consider has some similarities with the one studied in [7] (in a different context). The main difference here is that the a_j can depend on j. Nevertheless, we will use the same method and we introduce for all $j \in \mathbb{Z}$

$$\Xi_j(t) = U_j(t) + \frac{1}{\alpha}\dot{U}_j(t) \quad \text{where} \quad \alpha = \frac{1}{2}\min_{j\in\{1,\dots,n_0\}}(a_j).$$

We then obtain the following system of ODEs: for all $j \in \mathbb{Z}$ and $t \in (0, +\infty)$,

$$\begin{cases} \dot{U}_j(t) = \alpha(\Xi_j(t) - U_j(t)) \\ \dot{\Xi}_j(t) = (a_j - \alpha)(U_j(t) - \Xi_j(t)) + \dfrac{a_j}{\alpha} V_j(U_{j+1}(t) - U_j(t)). \end{cases} \tag{1.2}$$

Let us now give the assumptions on the functions V_j and the coefficients a_j:

(A1) (Regularity) For all $j \in \{1, ..., n_0\}$,

$$\begin{cases} V_j \text{ is continuous and non-negative.} \\ V_j \text{ is Lipschitz continuous and we denote by } L_j \text{ its Lipschitz constant.} \end{cases}$$

We denote by $L = \max_{j\in\{1,\dots,n_0\}} L_j$

(A2) (Monotonicity) For all $j \in \{1, ..., n_0\}$,

$$\begin{cases} V_j \text{ is non-decreasing.} \\ a_j \geq 4L. \end{cases}$$

(A3) (Upper bound) For all $j \in \{1, ..., n_0\}$,

$$\lim_{h\to+\infty} V_j(h) < +\infty. \tag{1.3}$$

We denote by $V_{max} = \max_j (||V_j||_\infty)$ and $h_0 = V_{max}/\alpha$.

(A4) (Lower bound) For all $j \in \{1, ..., n_0\}$,

$$V_j(h) = 0 \quad \text{for all } h \leq 2h_0.$$

Remark 1.1. *Conditions (A1)-(A3) are classical for the Bando model (see for example [1], [3]). Assumption (A4) appears for example in [3]. We note that the second condition in (A2) appears in [1] to get stability. Here, it allows us to show that $U_j \mapsto (a_j - \alpha)U_j + \frac{a_j}{\alpha} V_j (U_{j+1} - U_j)$ is non decreasing, and so the system (1.2) is monotone.*

(A5) (Periodicity of the type of drivers) For all $j \in \mathbb{Z}$,

$$a_{j+n_0} = a_j \quad \text{and} \quad V_{j+n_0} = V_j.$$

1.2 General system with n_0 types of drivers

As in [7, 9], we inject the system of (ODE) in a system of (PDE) by considering the functions

$$(u, \xi) = ((u_j(t,x))_{j\in\mathbb{Z}}, (\xi_j(t,x))_{j\in\mathbb{Z}})$$

which satisfies the following system of equations, for all $(t,x) \in (0,+\infty) \times \mathbb{R}$ and for all $j \in \mathbb{Z}$,

$$\begin{cases} \dfrac{\partial u_j}{\partial t}(t,x) = \alpha(\xi_j(t,x) - u_j(t,x)) \\ \dfrac{\partial \xi_j}{\partial t}(t,x) = (a_j - \alpha)(u_j(t,x) - \xi_j(t,x)) + \dfrac{a_j}{\alpha} V_j(u_{j+1}(t,x) - u_j(t,x)) \\ u_{j+n_0}(t,x) = u_j(t,x+1) \\ \xi_{j+n_0}(t,x) = \xi_j(t,x+1). \end{cases} \tag{1.4}$$

However, we are more interested in the rescaled system, defined by

$$u_j^\varepsilon(t,x) = \varepsilon u_j\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) \quad \text{and} \quad \xi_j^\varepsilon(t,x) = \varepsilon \xi_j\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right). \tag{1.5}$$

The function $(u^\varepsilon, \xi^\varepsilon) = ((u_j^\varepsilon(t,x))_{j\in\mathbb{Z}}, (\xi_j^\varepsilon(t,x))_{j\in\mathbb{Z}})$ satisfy the following Cauchy problem, for all $(t,x) \in (0,+\infty) \times \mathbb{R}$,

$$\begin{cases} \dfrac{\partial u_j^\varepsilon}{\partial t}(t,x) = \alpha \dfrac{\xi_j^\varepsilon(t,x) - u_j^\varepsilon(t,x)}{\varepsilon} \\ \dfrac{\partial \xi_j^\varepsilon}{\partial t}(t,x) = (a_j - \alpha)\dfrac{u_j^\varepsilon(t,x) - \xi_j^\varepsilon(t,x)}{\varepsilon} + \dfrac{a_j}{\alpha} V_j\left(\dfrac{u_{j+1}^\varepsilon(t,x) - u_j^\varepsilon(t,x)}{\varepsilon}\right) \\ u_{j+n_0}^\varepsilon(t,x) = u_j^\varepsilon(t,x+\varepsilon) \\ \xi_{j+n_0}^\varepsilon(t,x) = \xi_j^\varepsilon(t,x+\varepsilon), \end{cases} \tag{1.6}$$

completed with the initial condition

$$u_j^\varepsilon(0,x) = u_0\left(x + \frac{j\varepsilon}{n_0}\right) \quad \text{and} \quad \xi_j^\varepsilon(0,x) = \xi_0^\varepsilon\left(x + \frac{j\varepsilon}{n_0}\right). \tag{1.7}$$

We assume that the initial condition satisfies the following assumption,

(A0) (Gradient bound) There exist $k_0, K_0 > 0$ such that

$$\begin{cases} 0 < k_0 \le (u_0)_x \le K_0 \\ 0 < k_0 \le (\xi_0^\varepsilon)_x \le K_0. \end{cases}$$

We also assume that

$$0 \le \alpha(\xi_0^\varepsilon(x) - u_0(x)) \le \min\left(V_{max}\varepsilon, \alpha. \frac{u_0\left(x + \dfrac{\varepsilon}{n_0}\right) - u_0(x)}{2}\right) \quad \text{for all } x \in \mathbb{R}.$$

Remark 1.2. *Condition (A0) allows us to get that when the vehicles have enough space between them, the initial velocity of the vehicles is less than V_{max}. In the case where two cars are too close, condition (A0) ensures that the initial speed of each vehicle is bounded in a way to avoid collisions even in the worst case where the vehicle in front has completely stopped.*

The main purpose of this article is to prove that the viscosity solution of (1.6)-(1.7) converges uniformly on compact subsets of $(0,+\infty)\times\mathbb{R}$ as ε goes to 0, to the unique solution of the following problem

$$\begin{cases} u^0_t(t,x) = \bar{F}(u^0_x(t,x)) & \text{for } (t,x)\in(0,+\infty)\times\mathbb{R}, \\ u^0(0,x) = u_0(x) & \text{for } x\in\mathbb{R}, \end{cases} \tag{1.8}$$

where $\bar{F}$ has to be determined.

Theorem 1.3 (Homogenization of systems with n_0 types of drivers)**.** *Assume that (A1)-(A5) holds and that the initial datum u_0, ξ_0^ε satisfy (A0). Consider the solution $((u_j^\varepsilon)_{j\in\mathbb{Z}}, (\xi_j^\varepsilon)_{j\in\mathbb{Z}})$ of (1.6)-(1.7). Then, there exists a continuous function $\bar{F}:\mathbb{R}\to\mathbb{R}$ such that, for all $j\in\mathbb{Z}$, the functions u_j^ε and ξ_j^ε converge uniformly on compact subsets of $(0,+\infty)\times\mathbb{R}$ as ε goes to 0 to the unique viscosity solution u^0 of (1.8).*

Theorem 1.4 (Homogenization of systems with n_0 types of sensibilities and one OVF)**.** *Assume that (A1)-(A5) are satisfied and that the initial datum u_0, ξ_0^ε satisfy (A0). We also assume that $V_j = V$ for all $j\in\mathbb{N}$. Let us consider the solution $((u_j^\varepsilon)_{j\in\mathbb{Z}}, (\xi_j^\varepsilon)_{j\in\mathbb{Z}})$ of (1.6)-(1.7). Then the effective Hamiltonian $\bar{F}$ is given by*

$$\bar{F}(p) = V\left(\frac{p}{n_0}\right) \quad \text{for all } p\in\mathbb{R}^+. \tag{1.9}$$

1.3 Hull functions

We recall the notion of hull function (presented as in [7]) for the system (1.4) which is necessary for the proof of Theorem 1.3. It will allow us in particular to define the effective Hamiltonian $\bar{F}$.

Definition 1.5 (Hull function for system with n_0 types of drivers)**.** *Given $(V_j)_j$ and $(a_j)_j$ satisfying (A1)-(A5), $p\in\mathbb{R}^+$, and a real number $\lambda\in\mathbb{R}$, we say that a family of functions $((h_j)_{j\in\mathbb{Z}}, (g_j)_{j\in\mathbb{Z}})$ is a hull function for (1.4) if it satisfies for all $j\in\{1,...,n_0\}$ and for all $z\in\mathbb{R}$,*

$$\begin{cases} \lambda = \alpha(g_j - h_j) \\ h_{j+n_0}(z) = h_j(z+p) \\ h_{j+1}(z) \geq h_j(z) \\ h_j(z) = z + h_j(0) \end{cases} \quad \begin{cases} \lambda = (a_j-\alpha)(h_j-g_j) + \dfrac{a_j}{\alpha}V_j(h_{j+1}-h_j) \\ g_{j+n_0}(z) = g_j(z+p) \\ g_{j+1}(z) \geq g_j(z) \\ g_j(z) = z + g_j(0) \end{cases} \tag{1.10}$$

Remark 1.6. *The notion of hull functions is a little bit different from the one presented in [7]. This comes from the fact that our system is invariant by addition of constant while the one considered in [7] was invariant by addition of integer constant only. This allows us to show that $h_j' = 1$ and $g_j' = 1$ and so to get the special from for h_j and g_j.*

Theorem 1.7 (Effective Hamiltonian and hull functions)**.** *Assume (A1)-(A5) and let $p\in(0,+\infty)$. Then there exists a unique real λ for which there exists a hull function $((h_j)_j,(g_j)_j)$ satisfying (1.10). Moreover the real $\lambda = \bar{F}(p)$, seen as a function of p, is continuous in $(0,+\infty)$.*

Remark 1.8. *A simple computation gives us that*

$$\bar{F}(p) = V_j\left(h_{j+1}(0) - h_j(0)\right) \quad \forall j\in\mathbb{Z}. \tag{1.11}$$

1.4 Qualitative properties of the effective Hamiltonian

We have the following results concerning $\bar{F}$, and concerning the homogenized Hamilton-Jacobi equation (1.8).

Theorem 1.9 (Qualitative properties of $\bar{F}$)**.** *Assume (A1)-(A5). For any $p \in (0, +\infty)$, let $\bar{F}(p)$ denote the effective Hamiltonian given by Theorem 1.7. Then we have the following properties*

(i) (Lower boundary) *if $p \le 2h_0 n_0$, we have*

$$\bar{F}(p) = 0.$$

(ii) (Upper boundary)

$$\lim_{p \to +\infty} \bar{F}(p) = \min_{j \in \{1,\dots,n_0\}} (||V_j||_\infty).$$

(iii) (Monotonicity) *$\bar{F}$ is non-decreasing.*

Remark 1.10. *For example, an effective Hamiltonian can be of the form:*

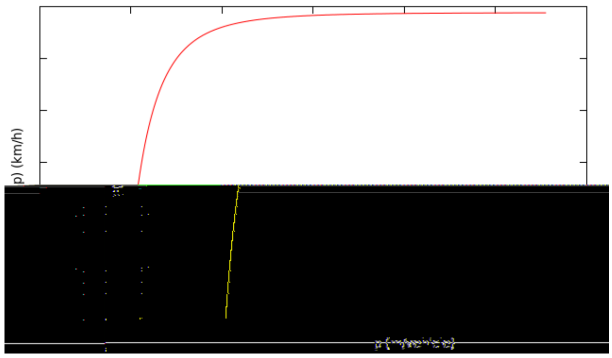

Figure 1: Schematic representation of the effective Hamiltonian.

Link with macroscopic models In the literature we can find different types of macroscopic models. But we will focus on the first order model LWR (Lighthill-Whitham-Richards) (for more information on the LWR model see for instance [17]), which is defined by

$$\partial_t \rho + \partial_y (\rho v(\rho)) = 0, \tag{1.12}$$

where $\rho(t, y)$ is the density of vehicles at the point $y \in \mathbb{R}$ (physical point on the road) at time $t \in (0, +\infty)$, and $v(\rho)$ is the average speed of vehicles. We call $f(\rho) = \rho v(\rho)$ the traffic flux. It can be remarked that (1.12) uses Eulerian coordinates (y is physical point on the road). However, it was proven by Wagner in [18] (for equations of gas dynamics) that the problem (1.12) is equivalent to

$$\partial_t s - \partial_x v^*(s) = 0, \tag{1.13}$$

where $s(t, x) = 1/\rho$ is the spacing between the vehicles, x stands for the vehicle x (seen as a continuous variable) and $v^*(s) = v(1/s)$. We can see that equation (1.13) uses Lagrangian coordinates. Moreover, if we denote by $u^0(t, x)$ the position of the x vehicle, we have that (1.13) is equivalent (see [14]) to

$$\partial_t u^0(t, x) = v^* \left(\partial_x u^0\right), \tag{1.14}$$

with $s(t,x) = \partial_x u^0(t,x)$. From this we can see that equation (1.8) is equivalent to a macroscopic model of traffic flow of the LWR type, with

$$v(\rho) := \bar{F}\left(\frac{1}{\rho}\right).$$

Using Theorem 1.9, we can see that the flux of the macroscopic model, $f(\rho) = \rho v(\rho)$, satisfies some of the properties presented in [17]:

1. f is a continuous function.
2. $f(0) = f(\rho_{max}) = 0$, with $\rho_{max} = \dfrac{n_0}{h_0}$.

1.5 Organisation of the article

In Section 2 we give some results concerning viscosity solutions for systems. In Section 3, we prove Theorem 1.3 assuming Theorem 1.7. In Section 4 we give the results concerning the existence of the hull functions.

2 Viscosity Solutions

This section is devoted to the definition and to useful results for viscosity solutions for systems like (1.4). The reader is referred to the user's guide of Crandall, Ishii, Lions [5] and the book of Barles [2] for an introduction to viscosity solutions and to [11, 12, 15] and references therein for results concerning viscosity solutions for weakly coupled systems.

2.1 Definitions

We consider for $0 < T \leq +\infty$ the following Cauchy problem, for $j \in \mathbb{Z}$, $t > 0$ and $x \in \mathbb{R}$,

$$\begin{cases} \dfrac{\partial u_j}{\partial t}(t,x) = & \alpha(\xi_j(t,x) - u_j(t,x)) \\ \dfrac{\partial \xi_j}{\partial t}(t,x) = & (a_j - \alpha)(u_j(t,x) - \xi_j(t,x)) + \dfrac{a_j}{\alpha} V_j(u_{j+1}(t,x) - u_j(t,x)) \\ u_{j+n_0}(t,x) = u_j(t,x+1) \\ \xi_{j+n_0}(t,x) = \xi_j(t,x+1), \end{cases} \tag{2.1}$$

with the initial condition

$$u_j(0,x) = u_0\left(x + \frac{j}{n_0}\right) \quad \text{and} \quad \xi_j(0,x) = \xi_0\left(x + \frac{j}{n_0}\right). \tag{2.2}$$

We recall the definition of the upper and lower semi-continuous envelopes, u^* and u_*, of a locally bounded function u,

$$u^*(t,x) = \limsup_{(\tau,y)\to(t,x)} u(\tau,y) \quad \text{and} \quad u_*(t,x) = \liminf_{(\tau,y)\to(t,x)} u(\tau,y).$$

Definition 2.1 (Viscosity Solutions)**.** *Let $T > 0$, $u_0 : \mathbb{R} \to \mathbb{R}$ and $\xi_0 : \mathbb{R} \to \mathbb{R}$ satisfying (A0'). For all j, let $u_j : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $\xi_j : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be upper semi-continuous (resp. lower semi-continuous) locally bounded functions. We set $\Omega = (0,T) \times \mathbb{R}$. Let us consider that $((u_j)_j, (\xi_j)_j)$ satisfies*

$$\forall j \in \mathbb{Z},\ \forall (t,x) \in \Omega,\ u_{j+n_0}(t,x) = u_j(t,x+1) \quad \textit{and} \quad \xi_{j+n_0}(t,x) = \xi_j(t,x+1).$$

-A function $((u_j)_j,(\xi_j)_j)$ is a sub-solution (resp. a super-solution) of (2.1) on Ω if for all $(t,x)\in\Omega$ and for any test function $\varphi\in C^1(\Omega)$ such that $u_j-\varphi$ attains a local maximum (resp. a local minimum) at the point (t,x), we have

$$\varphi_t(t,x)\le\alpha(\xi_j(t,x)-u_j(t,x))\ \text{(resp. }\ge), \tag{2.3}$$

and for all $(t,x)\in\Omega$, and any test function $\varphi\in C^1(\Omega)$ such that $\xi_j-\varphi$ attains a local maximum (resp. a local minimum) at the point (t,x), we have

$$\varphi_t(t,x)\le\quad(a_j-\alpha)(u_j(t,x)-\xi_j(t,x))+\tfrac{a_j}{\alpha}V_j(u_{j+1}(t,x)-u_j(t,x))\quad\text{(resp. }\ge) \tag{2.4}$$

-A function $((u_j)_j,(\xi_j)_j)$ is a sub-solution (resp. a super-solution) of (2.1)-(2.2) if $((u_j)_j,(\xi_j)_j)$ is a sub-solution (resp. a super solution) of (2.1) on Ω and if it satisfies moreover for all $x\in\mathbb{R}$, $j\in\{1,...,n_0\}$,

$$u_j(0,x)\le u_0\left(x+\frac{j}{n_0}\right)\ \text{(resp. }\ge)\quad\text{and}\quad\xi_j(0,x)\le\xi_0\left(x+\frac{j}{n_0}\right)\ \text{(resp. }\ge).$$

-A function $((u_j)_j,(\xi_j)_j)$ is a viscosity solution of (2.1) (resp. of (2.1)-(2.2)) if $((u_j^*)_j,(\xi_j^*)_j)$ is a sub-solution and $(((u_j)_*)_j,((\xi_j)_*)_j)$ is a super solution of (2.1) (resp. of (2.1)-(2.2)).

2.2 Results for viscosity solutions of (2.1)

Proposition 2.2 (Comparison Principle)**.** *Assume (A0) and (A1)-(A5). Let (u_j,ξ_j) and $(v_j,\zeta_j))$ be respectively a sub-solution and a super-solution of (2.1)-(2.2). We also assume that there is a constant $K>0$ such that for all $j\in\{1,...,n_0\}$ and for all $(t,x)\in[0;T]\times\mathbb{R}$, we have*

$$\begin{aligned}u_j(t,x)&\le u_j(0,x)+K(1+t),\quad \xi_j(t,x)\le\xi_j(0,x)+K(1+t)\\ -v_j(t,x)&\le -v_j(0,x)+K(1+t),\quad -\zeta_j(t,x)\le-\xi_j(0,x)+K(1+t).\end{aligned} \tag{2.5}$$

If

$$u_j(0,x)\le v_j(0,x)\quad\text{and}\quad\xi_j(0,x)\le\zeta_j(0,x)\quad\text{for all } x\in\mathbb{R},\ j\in\mathbb{Z},$$

then

$$u_j(t,x)\le v_j(t,x)\quad\text{and}\quad\xi_j(t,x)\le\zeta_j(t,x)\quad\text{for all } x\in\mathbb{R},\ j\in\mathbb{Z},\ t\in[0;T].$$

Proof of Proposition 2.2.

This proof is similar to the one of Proposition 2.2 in [7], because the system (2.1) is monotone as the one studied in [7] thanks to assumption (A2), so we skip it.

□

We now give a comparison principle on bounded sets, to do this, we define, for a given point $(t_0,x_0)\in(0,T)\times\mathbb{R}$ and for all $r,R>0$, the set

$$\mathcal{Q}_{r,R}=(t_0-r,t_0+r)\times(y_0-R,y_0+R).$$

Proposition 2.3 (Comparison principle on bounded sets)**.** *Assume (A1)-(A5). Let $((u_j)_j,(\xi_j)_j)$ (resp. $((v_j)_j,(\zeta_j)_j)$) be a sub-solution (resp. a super-solution) of (2.1) on the open set $\mathcal{Q}_{r,R}\subset(0,T)\times\mathbb{R}$. Assume also that for all $j\in\{1,...,n_0\}$,*

$$u_j\le v_j\quad\text{and}\quad\xi_j\le\zeta_j\quad\text{on } \bar{\mathcal{Q}}_{r,R}\setminus\mathcal{Q}_{r,R},$$

then

$$u_j\le v_j\quad\text{and}\quad\xi_j\le\zeta_j\quad\text{on } \mathcal{Q}_{r,R}\text{ for } j\in\{1,...,n_0\}.$$

We now turn to the existence of a solution for equation (2.1). To do this we will use the following lemma.

Lemma 2.4 (Existence of Barriers)**.** *Assume (A0) and (A1)-(A3). There exist a constant* $K_1 > 0$ *such that*

$$((u_j^+(t,x))_j, (\xi_j^+(t,x))_j) = \left(\left(u_0\left(x + \frac{j}{n_0}\right) + K_1 t \right)_j, \left(\xi_0\left(x + \frac{j}{n_0}\right) + K_1 t \right)_j \right),$$

and

$$((u_j^-(t,x))_j, (\xi_j^-(t,x))_j) = \left(\left(u_0\left(x + \frac{j}{n_0}\right) - K_1 t \right)_j, \left(\xi_0\left(x + \frac{j}{n_0}\right) - K_1 t \right)_j \right),$$

are respectively super and sub-solutions of (2.1)-(2.2) for all $T > 0$*. Moreover, the constant* K_1 *can be chosen to be*

$$K_1 = \max_{j \in \{1,\dots,n_0\}} (a_j).\frac{V_{max}}{\alpha}.$$

Proof. Let us prove that $((u_j^+(t,x))_j, (\xi_j^+(t,x)_j))$ is a super-solution of (2.1)-(2.2).

First, using (A0) with $\varepsilon = 1$ we have for all $j \in \{1, ..., n_0\}$,

$$\alpha(\xi_j^+(t,x) - u_j^+(t,x)) = \alpha\left(\xi_0\left(x + \frac{j}{n_0}\right) - u_0\left(x + \frac{j}{n_0}\right) \right) \le V_{max} \le K_1,$$

and

$$(a_j - \alpha)(u_j^+(t,x) - \xi_j^+(t,x)) + \frac{a_j}{\alpha} V_j(u_{j+1}^+(t,x) - u_j^+(t,x)) \le \frac{\max(a_j)}{\alpha} V_{max} \le K_1, \qquad (2.6)$$

where we have used (A0) and the fact that for all j, $||V_j||_\infty \le V_{max}$.

□

By applying Perron's method, joint to the comparison principle, we get the following result.

Theorem 2.5 (Existence and uniqueness of viscosity solutions for (2.1))**.** *Assume (A0) and (A1)-(A5). Then there exists a unique solution* $((u_j)_j, (\xi_j)_j)$ *of (2.1)-(2.2). Moreover, the functions* u_j, ξ_j *are continuous for all* $j \in \mathbb{Z}$*.*

We now prove that the cars remain in ordered during the evolution.

Theorem 2.6 (Ordering of the cars)**.** *Assume (A0) and (A1)-(A5). Let* $((u_j)_j, (\xi_j)_j)$ *be a solution of (2.1). Then* u_j *and* ξ_j *are non-decresing with respect to* j*.*

In order to do the proof of this theorem we need the following lemma.

Lemma 2.7 (Bound on time-derivative)**.** *Assume (A1)-(A5). Let* $((u_j)_j, (\xi_j)_j)$ *be a solution of (1.4), with initial condition* $((u_j(0,x))_j, (\xi_j(0,x))_j)$ *satisfying (A0). Then for all* $(t,x) \in [0,T] \times \mathbb{R}$ *and for all* $j \in \mathbb{Z}$*,*

$$0 \le \xi_j(t,x) - u_j(t,x) \le \frac{V_{max}}{\alpha}.$$

Proof. Since $\xi_j - u_j$ is periodic in j it is sufficient to do this proof for $j \in \{1, ..., n_0\}$. We will only do the proof for the upper bound since the proof for the lower bound is similar.

Step 1: test function. We introduce

$$M = \sup_{(t,x)\in[0,T]\times\mathbb{R}} \max_{j\in\{1,\ldots,n_0\}} \left(\xi_j(t,x) - u_j(t,x) - \frac{V_{max}}{\alpha}\right).$$

We want to prove that $M \le 0$. To do this, we argue by contradiction and we assume that $M > 0$. We define the following function, with $\varepsilon, \eta, \gamma > 0$ small parameters,

$$\varphi(t,x,y,j) = \xi_j(t,x) - u_j(t,y) - \frac{|x-y|^2}{\varepsilon^2} - \frac{\eta}{T-t} - \frac{V_{max}}{\alpha} - \gamma|x|^2.$$

We can see that the function $\varphi(t,x,y,j)$ reaches a maximum at a finite point $(\bar{t},\bar{x},\bar{y},\bar{j})$ thanks to the existence of barriers (Lemma 2.4). By classical arguments, we have,

$$\begin{cases} M_{\varepsilon,\eta,\gamma} = \varphi(\bar{t},\bar{x},\bar{y},\bar{j}) \ge \dfrac{M}{2} \text{ for } \gamma \text{ and } \eta \text{ small enough.} \\ |\bar{x}-\bar{y}| \to 0 \text{ as } \varepsilon \to 0. \\ \gamma|\bar{x}| \to 0 \text{ as } \alpha \to 0. \end{cases} \tag{2.7}$$

Step 2: $\bar{t} > 0$ for ε small enough. By contradiction, let us assume $\bar{t} = 0$. Then using the fact that $M_{\varepsilon,\eta,\gamma} \ge M/2 > 0$, we get

$$\begin{aligned} 0 < M_{\varepsilon,\eta,\gamma} \le\ & \xi_{\bar{j}}(0,\bar{x}) - u_{\bar{j}}(0,\bar{y}) - \frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} - \frac{\eta}{T} - \gamma|\bar{x}|^2 - \frac{V_{max}}{\alpha} \\ \Rightarrow \frac{\eta}{T} <\ & u_{\bar{j}}(0,\bar{x}) - u_{\bar{j}}(0,\bar{y}) + \xi_{\bar{j}}(0,\bar{x}) - u_{\bar{j}}(0,\bar{x}) - \frac{V_{max}}{\alpha} \\ \Rightarrow \frac{\eta}{T} <\ & K_0|\bar{x}-\bar{y}| + \frac{V_{max}}{\alpha} - \frac{V_{max}}{\alpha} \le K_0|\bar{x}-\bar{y}|, \end{aligned} \tag{2.8}$$

where we have used for the third line assumption (A0). This is a contradiction for ε small enough.

Step 3: viscosity inequalities. We classically do a duplication of the time variable and passing to the limit, we get that there are real numbers $a, b \in \mathbb{R}$ such that

$$a - b = \frac{\eta}{(T-\bar{t})^2},$$

and

$$\begin{aligned} & a \le (a_{\bar{j}} - \alpha)(u_{\bar{j}}(\bar{t},\bar{x}) - \xi_{\bar{j}}(\bar{t},\bar{x})) + \frac{a_{\bar{j}}}{\alpha} V_{\bar{j}} \left(u_{\bar{j}+1}(\bar{t},\bar{x}) - u_{\bar{j}}(\bar{t},\bar{x})\right) \\ & b \ge \alpha \left(\xi_{\bar{j}}(\bar{t},\bar{y}) - u_{\bar{j}}(\bar{t},\bar{y})\right). \end{aligned} \tag{2.9}$$

Step 4: passing to the limit. Subtracting the two previous inequalities we obtain

$$\begin{aligned} \frac{\eta}{T^2} \le\ & (a_{\bar{j}} - \alpha)\left(u_{\bar{j}}(\bar{t},\bar{x}) - \xi_{\bar{j}}(\bar{t},\bar{x})\right) + \alpha\left(u_{\bar{j}}(\bar{t},\bar{y}) - \xi_{\bar{j}}(\bar{t},\bar{y})\right) + \frac{a_{\bar{j}}}{\alpha} V_{\bar{j}} \left(u_{\bar{j}+1}(\bar{t},\bar{x}) - u_{\bar{j}}(\bar{t},\bar{x})\right) \\ \le\ & \alpha\left(u_{\bar{j}}(\bar{t},\bar{y}) - u_{\bar{j}}(\bar{t},\bar{x}) + \xi_{\bar{j}}(\bar{t},\bar{x}) - \xi_{\bar{j}}(\bar{t},\bar{y})\right) + a_{\bar{j}}(u_{\bar{j}}(\bar{t},\bar{x}) - \xi_{\bar{j}}(\bar{t},\bar{x})) + a_{\bar{j}}\frac{V_{max}}{\alpha}. \end{aligned} \tag{2.10}$$

Using that $a_{\bar{j}}\dfrac{V_{max}}{\alpha} \le a_{\bar{j}}(\xi_{\bar{j}}(\bar{t},\bar{x}) - u_{\bar{j}}(\bar{t},\bar{y}))$, we get that

$$\frac{\eta}{T^2} \le \alpha\left(u_{\bar{j}}(\bar{t},\bar{y}) - u_{\bar{j}}(\bar{t},\bar{x}) + \xi_{\bar{j}}(\bar{t},\bar{x}) - \xi_{\bar{j}}(\bar{t},\bar{y})\right) + a_{\bar{j}}(u_{\bar{j}}(\bar{t},\bar{y}) - u_{\bar{j}}(\bar{t},\bar{x})).$$

Sending $\varepsilon \to 0$, we get $\dfrac{\eta}{t^2} \le 0$, which is a contradiction.

□

We now turn to the proof of Theorem 2.6.

Proof of Theorem 2.6.

Case 1: $k_0 \geq 2h_0n_0$. In this case we have for all $j \in \mathbb{Z}$ and $y \in \mathbb{R}$,

$$\frac{2V_{max}}{\alpha} = 2h_0 \leq u_0\left(y + \frac{j+1}{n_0}\right) - u_0\left(y + \frac{j}{n_0}\right)$$

Now, we would like to prove that $u_j(t,y) < u_{j+1}(t,y) - h_0$. We argue by contradiction, let us assume that there exists a time

$$t^* = \inf\left\{t, \text{ s.t } \exists i \in \mathbb{Z},\ y \in \mathbb{R} \quad \text{s.t} \quad u_i(t,y) = u_{i+1}(t,y) - h_0\right\}.$$

Let us consider $y \in \mathbb{R}$ and $i \in \mathbb{Z}$ such that $u_i(t^*,y) = u_{i+1}(t^*,y) - 2h_0$. By continuity, there exists a time $t_0 \in [0,t^*)$ such that

$$u_i(t_0,y) = u_{i+1}(t_0,y) - 2h_0 \quad \text{and} \quad u_{i+1}(t,y) - u_i(t,y) \in [h_0, 2h_0] \quad \text{for all } t \in [t_0,t^*].$$

Let us now see the equation satisfied by (u_i,ξ_i) for $t \in [t_0,t^*]$, using that $V_i(u_{i+1}(t,y) - u_i(t,y)) = 0$,

$$\begin{cases} (u_i)_t(t,y) = \alpha(u_i(t,y) - \xi_i(t,y)) \\ (\xi_i)_t(t,y) = (a_i - \alpha)(u_i(t,y) - \xi_i(t,y)) \\ u_i(t_0,y) = u_{i+1}(t_0,y) - 2h_0 \\ \xi_i(t_0,y) \leq u_{i+1}(t_0,y) - 2h_0 + \dfrac{V_{max}}{\alpha}. \end{cases} \tag{2.11}$$

The last inequality is justified by Lemma 2.7. We now construct a super-solution for this system by considering

$$\begin{cases} \bar{u}_i(t,y) = \frac{V_{max}}{a_i}(1 - e^{-a_i(t-t_0)}) + u_{i+1}(t_0,y) - 2h_0 \\ \bar{\xi}_i(t,y) = \bar{u}_i(t,y) + \frac{1}{\alpha}(\bar{u}_i)_t(t,y), \end{cases}$$

with the initial condition

$$\begin{cases} \bar{u}_i(t_0,y) = u_{i+1}(t_0,y) - 2h_0 \\ \bar{\xi}_i(t_0,y) = u_{i+1}(t_0,y) - 2h_0 + \frac{V_{max}}{\alpha}. \end{cases}$$

Since $t_0 < t^*$, we have

$$\begin{aligned} \bar{u}_i(t^*,y) < {} & \frac{V_{max}}{\alpha} - h_0 + u_{i+1}(t_0,y) - h_0 \\ = {} & u_{i+1}(t_0,y) - h_0 \\ \leq {} & u_{i+1}(t^*,y) - h_0, \end{aligned} \tag{2.12}$$

where we used for the first line the fact that $1 - e^{a_i(t^*-t_0)} < 1$ and that $\alpha \leq a_i$, for the second line the fact that $h_0 = V_{max}/\alpha$, and for the third line the fact that $t^* > t_0$ and Lemma 2.7 which implies that the u_j are non-decreasing in t. Using the comparison principle for (2.11) yields

$$u_i(t^*,y) \leq \bar{u}_i(t^*,y) < u_{i+1}(t^*,y) - h_0. \tag{2.13}$$

This is a contradiction with the definition of t^*. Therefore, we have for all $j \in \{1, ..., n_0\}$, for all $(t,x) \in [0,T] \times \mathbb{R}$, that

$$u_j(t,x) < u_{j+1}(t,x) - h_0.$$

Now from Lemma 2.7 we know that for all $(t,x) \in (0,+\infty)\times\mathbb{R}$ and for all $j \in \mathbb{Z}$,

$$\begin{cases} V_{max} \geq \alpha(\xi_j(t,x) - u_j(t,x)) \geq 0 \\ V_{max} \geq \alpha(\xi_{j+1}(t,x) - u_{j+1}(t,x)) \geq 0, \end{cases}$$

from which we can easily deduce that for all $j \in \mathbb{Z}$, for all $(t,x) \in [0,T]\times\mathbb{R}$, we have

$$\xi_{j+1}(t,x) \geq \xi_j(t,x).$$

Case 2: $0 < k_0 \leq 2h_0 n_0$. We use the following lemma:

Lemma 2.8. *Assume (A1)-(A5), let $i \in \mathbb{Z}$, $y \in \mathbb{R}$, and let $2h_0 - k_0/n_0 \geq \delta > 0$, also let us assume that*

$$u_i(0,y) = u_{i+1}(0,y) - (2h_0 - \delta) \quad \text{and} \quad 0 \leq \xi_i(0,y) - u_i(0,y) \leq \frac{2h_0 - \delta}{2}, \tag{2.14}$$

then we have

$$u_i(t,y) \leq u_{i+1}(t,y) \quad \text{and} \quad \xi_i(t,y) \leq \xi_{i+1}(t,y),$$

for all time $t \in [0,T]$ such that

$$u_{i+1}(t,y) - u_i(t,y) \leq 2h_0.$$

Proof. Let us denote by $\hat{t}$, the time

$$\hat{t} = \inf\{t \text{ s.t. } u_{i+1}(t,y) - u_i(t,y) = 2h_0\},$$

then for all $t \in [0,\hat{t}]$, $u_{i+1}(t,y) - u_i(t,y) \leq 2h_0$, and (u_i,ξ_i) is solution to the following system, for $t \in [0,\hat{t}]$

$$\begin{cases} (u_i)_t = \alpha(\xi_i - u_i) \\ (\xi_i)_t = (a_i - \alpha)(u_i - \xi_i) \\ u_i(0,y) = u_{i+1}(0,y) - (2h_0 - \delta) \\ \xi_i(0,y) \leq u_{i+1}(0,y) - 2h_0 + \delta + \dfrac{2h_0 - \delta}{2}, \end{cases}$$

We can construct a super-solution of this system by considering

$$\begin{cases} \bar{u}_i(t,y) = \alpha \dfrac{2h_0 - \delta}{2a_i}(1 - e^{-a_i t}) + u_{i+1}(0,y) - 2h_0 + \delta \\ \bar{\xi}_i(t,y) = \bar{u}_i(t,y) + \dfrac{1}{\alpha}(\bar{u}_i)_t(t,y). \end{cases} \tag{2.15}$$

Therefore, for all $t \in [0,\hat{t}]$, we have

$$\begin{aligned} u_{i+1}(t,y) - u_i(t,y) \geq \quad & u_{i+1}(0,y) - u_i(t,y) \\ \geq \quad & u_{i+1}(0,y) - \bar{u}_i(t,y) \\ \geq \quad & 2h_0 - \delta - \alpha\frac{2h_0 - \delta}{2a_i} \\ \geq \quad & 2h_0 - \delta - \frac{2h_0 - \delta}{2} \\ \geq \quad & \frac{2h_0 - \delta}{2} \geq 0, \end{aligned} \tag{2.16}$$

where we have used for the first line the fact that u_{i+1} is non-decreasing in time (see Lemma 2.7), for the second line a comparison between u_i and $\bar{u}_i$, for the third line the definition of $\bar{u}_i(t,y)$ and the fact that $1-e^{-a_i t} \leq 1$, and for the fourth line the fact that $\alpha \leq a_i$. Similarly, we have

$$\begin{aligned} \xi_{i+1}(t,y)-\xi_i(t,y) \geq &\ u_{i+1}(t,y)-\bar{\xi}_i(t,y) \\ \geq &\ u_{i+1}(t,y)-\bar{u}_i(t,y)-\frac{1}{\alpha}(\bar{u}_i)_t(t,y) \\ \geq &\ \frac{2h_0-\delta}{2}-\frac{2h_0-\delta}{2} \geq 0, \end{aligned} \tag{2.17}$$

where we have used for the first line Lemma 2.7 and the fact that $\bar{\xi}_i \geq \xi_i$, for the second line the definition of $\bar{\xi}$.

□

Using this lemma, we deduce that in the case where there exist $i \in \mathbb{Z}$ and $y \in \mathbb{R}$ such that

$$\frac{u_{i+1}(0,y)-u_i(0,y)}{2} \leq \frac{V_{max}}{\alpha} = h_0, \tag{2.18}$$

we have

$$u_i(t,y) \leq u_{i+1}(t,y) \quad \text{and} \quad \xi_i(t,y) \leq \xi_{i+1}(t,y),$$

for all time $t \in [0,t^*]$, where t^* is defined by

$$t^* = \inf\{t,\ u_{i+1}(t,y)-u_i(0,y) > 2h_0\}.$$

We can then use the same proof as in Case 1 to deduce that

$$u_{i+1}(t,y) \geq u_i(t,y)+h_0 \quad \text{and} \quad \xi_{i+1}(t,y) \geq \xi_i(t,y) \quad \text{for all } t \geq t^*.$$

□

3 Convergence

This section contains the proof of the main homogenization result (Theorem 1.3). This proof relies on the existence of hull functions and some properties of the effective Hamiltonian.

We recall two lemmas, necessary for the proof of Theorem 1.3. We begin by a first result which is a direct consequence of Perron's method and Lemma 2.4 (with a rescaling in ε).

Lemma 3.1 (Barriers uniform in ε)**.** *Assume (A1)-(A5) and (A0). Then there is a constant $C>0$, such that for all $\varepsilon>0$, the solution $((u_j^\varepsilon)_j,(\xi_j^\varepsilon)_j)$ of (1.6)-(1.7) satisfies for all $t>0$ and $x \in \mathbb{R}$,*

$$\left|u_j^\varepsilon(t,x)-u_0\left(x+\frac{j\varepsilon}{n_0}\right)\right| \leq Ct \quad \textit{and} \quad \left|\xi_j^\varepsilon(t,x)-\xi_0\left(x+\frac{j\varepsilon}{n_0}\right)\right| \leq Ct.$$

Lemma 3.2 (ε-bounds on the gradient)**.** *Assume (A1)-(A5) and (A0). Then, the solution $((u_j^\varepsilon)_j,(\xi_j^\varepsilon)_j)$ of (1.6)-(1.7) satisfies for all $t>0$, $x \in \mathbb{R}$,$z>0$ and $j \in \mathbb{Z}$,*

$$zk_0 \leq u_j^\varepsilon(t,x+z)-u_j^\varepsilon(t,x) \leq zK_0, \tag{3.1}$$

and

$$zk_0 \leq \xi_j^\varepsilon(t,x+z)-\xi_j^\varepsilon(t,x) \leq zK_0. \tag{3.2}$$

Proof. We prove the lower bound (the proof for the upper bound is similar). Using assumption (A0), we get that for all $j \in \mathbb{Z}$, for all $x \in \mathbb{R}$, z>0,

$$u_j^\varepsilon(0, x+z) = u_0\left(x + z + \frac{j\varepsilon}{n_0}\right) \geq u_0\left(x + \frac{j\varepsilon}{n_0}\right) + zk_0 \geq u_j^\varepsilon(0,x) + zk_0, \tag{3.3}$$

and

$$\xi_j^\varepsilon(0, x+z) \geq \xi_j^\varepsilon(0,x) + zk_0.$$

From the form of system (1.6), we know that the equation is invariant by addition of constants to the solutions. For this reason the solution associated to the initial data $((u_j^\varepsilon(0,x)+zk_0)_j, (\xi_j^\varepsilon(0,x)+zk_0)_j)$ is $((u_j^\varepsilon(t,x)+zk_0)_j, (\xi_j^\varepsilon(t,x)+zk_0)_j)$. We can also see that the equation is invariant by space translations. Therefore the solution associated to the initial data $((u_j(0,x+z))_j, (\xi_j^\varepsilon(0,x+z))_j)$ is $((u_j(t,x+z))_j, (\xi_j^\varepsilon(t,x+z))_j)$. Finally, from (3.3) and from the comparison principle (Proposition 2.2), we get

$$u_j^\varepsilon(t, x+z) \geq u_j(t,x) + zk_0 \quad \text{and} \quad \xi_j^\varepsilon(t,x+z) \geq \xi_j^\varepsilon(t,x) + zk_0.$$

□

Proof of Theorem 1.3.
Since, for all $j \in \mathbb{Z}$, $u_{j+n_0}^\varepsilon(t,x) = u_j^\varepsilon(t, x+\varepsilon)$ and $\xi_{j+n_0}^\varepsilon(t,x) = \xi_j(t,x+\varepsilon)$ it is sufficient to do this proof for $j \in \{1, ..., n_0\}$.

We introduce for all $j \in \{1, ..., n_0\}$,

$$\bar{u}_j(t,x) = \limsup_{\varepsilon \to 0, (t',x') \to (t,x)} u_j^\varepsilon(t',x') \quad \text{and} \quad \bar{\xi}_j(t,x) = \limsup_{\varepsilon \to 0, (t',x') \to (t,x)} \xi_j^\varepsilon(t',x'),$$

$$\underline{u}_j(t,x) = \liminf_{\varepsilon \to 0, (t',x') \to (t,x)} u_j^\varepsilon(t',x') \quad \text{and} \quad \underline{\xi}_j(t,x) = \liminf_{\varepsilon \to 0, (t',x') \to (t,x)} \xi_j^\varepsilon(t',x').$$

Thanks to Lemma 3.1 we know that this functions are well defined. We also introduce

$$\bar{v} = \max_{j \in \{1,...,n_0\}} \max(\bar{u}_j, \bar{\xi}_j) \quad \text{and} \quad \underline{v} = \min_{j \in \{1,...,n_0\}} \min(\underline{u}_j, \underline{\xi}_j). \tag{3.4}$$

Using the two previous lemmas we get that the function $w = \bar{v}, \underline{v}$, satisfies for all $t > 0$ and $x, x' \in \mathbb{R}$, $x \leq x'$,

$$\begin{aligned} &|w(t,x) - u_0(x)| \leq Ct, \\ &k_0|x - x'| \leq w(t,x') - w(t,x) \leq K_0|x-x'|. \end{aligned} \tag{3.5}$$

We want to prove that $\bar{v}$ is a sub-solution of (1.8) and that $\underline{v}$ is a super-solution of (1.8). Indeed, in this case, the comparison principle will imply that $\bar{v} \leq \underline{v}$. But by construction $\underline{v} \leq \bar{v}$, hence $\underline{v} = \bar{v} = u^0$, the unique solution of (1.8). This implies that for all $j \in \{1, ..., n_0\}$, $\bar{u}_j = \underline{u}_j = \bar{\xi}_j = \underline{\xi}_j = u^0$ and so u_j^ε and ξ_j^ε converge locally uniformly to u^0.

To prove that $\bar{v}$ is a sub-solution of (1.8), we argue by contradiction, we assume that there is a point $(\bar{t}, \bar{x}) \in \mathbb{R}^+ \times \mathbb{R}$ and a test function $\phi \in C^1$ such that

$$\begin{cases} \bar{v}(\bar{t},\bar{x}) = \phi(\bar{t},\bar{x}) & & \\ \bar{v} \leq \phi & \text{on } Q_{r,2r}(\bar{t},\bar{x}) & \text{with } r > 0 \\ \bar{v} \leq \phi - 2\eta & \text{on } Q_{r,2r}(\bar{t},\bar{x}) \backslash Q_{r,r}(\bar{t},\bar{x}) & \text{with } \eta > 0 \\ \phi_t(\bar{t},\bar{x}) = \bar{F}(\phi_x(\bar{t},\bar{x})) + \theta, & \text{with } \theta > 0. & \end{cases} \tag{3.6}$$

We define $p = \phi_x(\bar{t}, \bar{x})$ that according to (3.5) satisfies

$$0 < k_0 \leq p \leq K_0.$$

Using Theorem 1.7, we define the hull functions $((h_j)_j, (g_j)_j)$ associated to p such that

$$\lambda = \bar{F}(p)$$

We now apply the perturbed test function method introduced by Evans [6] in terms here of hull functions instead of correctors. Let us consider the following perturbed test functions for $j \in \{1, ..., n_0\}$,

$$\phi_j^\varepsilon(t,x) = \varepsilon h_j\left(\frac{\phi(t,x)}{\varepsilon}\right) = \phi(t,x) + \varepsilon h_j(0) \quad \text{and} \quad \psi_j^\varepsilon(t,x) = \varepsilon g_j\left(\frac{\phi(t,x)}{\varepsilon}\right) = \phi(t,x) + \varepsilon g_j(0).$$

We define the family of test functions $(\phi_j^\varepsilon)_{j\in\mathbb{Z}}$ and $(\psi_j^\varepsilon)_{j\in\mathbb{Z}}$ by using the relation

$$\phi_{j+kn_0}^\varepsilon(t,x) = \phi_j^\varepsilon(t, x+\varepsilon k) \quad \text{and} \quad \psi_{j+kn_0}^\varepsilon(t,x) = \psi_j^\varepsilon(t, x+\varepsilon k).$$

We first want to prove that $((\phi_j^\varepsilon)_j, (\psi_j^\varepsilon)_j)$ is a super-solution of (1.6) in a neighbourhood of $(\bar{t}, \bar{x})$.

To do this, we simply check the equations satisfied by the perturbed test functions, we denote by $z = \frac{\phi(t,x)}{\varepsilon}$ to simply the notations. For $j \in \{1, ..., n_0\}$, we have

$$\begin{aligned}
(\phi_j^\varepsilon)_t(t,x) = \;& \phi_t(t,x) + \alpha(g_j(z) - h_j(z)) - \alpha(g_j(z) - h_j(z)) \\
= \;& \frac{\alpha}{\varepsilon}(\psi_j^\varepsilon(t,x) - \phi_j^\varepsilon(t,x)) + (\phi_t(t,x) - \lambda) \\
= \;& \frac{\alpha}{\varepsilon}(\psi_j^\varepsilon(t,x) - \phi_j^\varepsilon(t,x)) + \big(\phi_t(t,x) - \phi_t(\bar{t},\bar{x}) + \theta\big) \\
\geq \;& \frac{\alpha}{\varepsilon}(\psi_j^\varepsilon(t,x) - \phi_j^\varepsilon(t,x)),
\end{aligned}$$

where we have used the equations satisfied by the hull functions for the second line, the definition of λ for the third line. For the fourth line we have used the fact that for $r > 0$ small enough, we have $\big(\phi_t(t,x) - \phi_t(\bar{t},\bar{x}) + \frac{\theta}{2}\big) \geq 0$, because $\theta > 0$ and ϕ is C^1. Similarly, we have

$$\begin{aligned}
(\psi_j^\varepsilon)_t(t,x) = \;& \phi_t(t,x) \\
= \;& (a_j - \alpha)(h_i(z) - g_j(z)) + \frac{a_j}{\alpha} V_j(h_{j+1}(z) - h_j(z)) - \lambda + \phi_t(t,x) \\
\geq \;& \frac{(a_j - \alpha)}{\varepsilon}(\phi_j^\varepsilon(t,x) - \psi_j^\varepsilon(t,x)) + \frac{a_j}{\alpha} V_j\left(\frac{\phi_{j+1}^\varepsilon(t,x) - \phi_j^\varepsilon(t,x)}{\varepsilon}\right) \\
& + \left(\phi_t(t,x) - \phi_t(\bar{t},\bar{x}) + \frac{\theta}{2}\right) \\
& + \frac{\theta}{2} + \frac{a_j}{\alpha}\left[V_j(h_{j+1}(z) - h_j(z)) - V_j\left(\frac{\phi_{j+1}^\varepsilon(t,x) - \phi_j^\varepsilon(t,x)}{\varepsilon}\right)\right].
\end{aligned}$$

It is then enough to prove that

$$\frac{\theta}{2} + \frac{a_j}{\alpha}\left[V_j(h_{j+1}(z) - h_j(z)) - V_j\left(\frac{\phi_{j+1}^\varepsilon(t,x) - \phi_j^\varepsilon(t,x)}{\varepsilon}\right)\right] \geq 0.$$

If $j+1 \in \{1, ..., n_0\}$, then,

$$h_j(z) = \frac{\phi_j^\varepsilon(t,x)}{\varepsilon} \quad \text{and} \quad h_{j+1}(z) = \frac{\phi_{j+1}^\varepsilon(t,x)}{\varepsilon},$$

and the result is trivial.

If $j+1 \notin \{1, ..., n_0\}$ then there is a $\tilde{k} \in \{1, ..., n_0\}$ such that $k = \tilde{k} + l_k n_0$,

$$h_{j+1}(z) = h_{1+n_0}(z) = h_1(z+p) = \frac{\phi(t,x)}{\varepsilon} + p + h_1(0),$$

$$\begin{aligned} \phi_{j+1}^\varepsilon(t,x) = \phi_{n_0+1}^\varepsilon(t,x) = & \quad \phi_1^\varepsilon(t, x+\varepsilon) = \phi(t, x+\varepsilon) + \varepsilon h_1(0). \\ = & \quad \phi(t,x) + \varepsilon p + \varepsilon h_1(0) + +o_\varepsilon(\varepsilon). \end{aligned}$$

This implies that

$$\begin{aligned} \frac{\phi_{j+1}^\varepsilon(t,x)}{\varepsilon} &= \frac{\phi(t,x)}{\varepsilon} + h_1(0) + o_\varepsilon(1) \\ &= h_{j+1}(z) + o_\varepsilon(1). \end{aligned}$$

This allows us to see that for $r > 0$ small enough, we get

$$\frac{\theta}{2} + \frac{a_j}{\alpha}\left[V_j(h_{j+1}(z) - h_j(z)) - V_j\left(\frac{\phi_{j+1}^\varepsilon(t,x) - \phi_j^\varepsilon(t,x)}{\varepsilon}\right)\right] \geq 0.$$

Getting a contradiction. By definition: $\phi_j^\varepsilon \to \phi$ and $\psi_j^\varepsilon \to \phi$ as $\varepsilon \to 0$. Moreover, $\bar{u}_j \leq \bar{v} \leq \phi - 2\eta$ on $Q_{r,2r}(\bar{t},\bar{x}) \backslash Q_{r,r}(\bar{t},\bar{x})$ therefore, for ε small enough

$$u_j^\varepsilon \leq \phi_j^\varepsilon - \eta \leq \phi_j^\varepsilon - \eta \text{ on } Q_{r,2r}(\bar{t},\bar{x}) \backslash Q_{r,r}(\bar{t},\bar{x}).$$

Similarly, we have

$$\xi_j^\varepsilon \leq \psi_j^\varepsilon - \eta \leq \psi_j^\varepsilon - \eta \text{ on } Q_{r,2r}(\bar{t},\bar{x}) \backslash Q_{r,r}(\bar{t},\bar{x}).$$

Using the comparison principle on bounded sets for (1.6), we get

$$u_j^\varepsilon \leq \phi_j^\varepsilon - \eta \text{ and } \xi_j^\varepsilon \leq \psi_j^\varepsilon - \eta \text{ on } Q_{r,r}(\bar{t},\bar{x}). \tag{3.7}$$

Passing to the limit as $\varepsilon \to 0$, we get $\bar{v} \leq \phi - \eta$ on $Q_{r,r}(\bar{t},\bar{x})$ and this contradicts the fact that $\bar{v}(\bar{t},\bar{x}) = \phi(\bar{t},\bar{x})$.

Therefore $\bar{v}$ is a sub-solution of (1.8) on $(0,+\infty)\times\mathbb{R}$. Similarly, $\underline{v}$ is a super-solution of the same equation. Therefore, $\underline{v} = \bar{v} = u^0$ and u_j and ξ_j converge locally uniformly to u^0 for $j \in \{1, ..., n_0\}$. □

4 Ergodicity and construction of hull functions

In this section, we construct the hull functions for (2.1). The construction follows the one of [7] but we use here the fact that the system is invariant by addition of constants. This allows us to get the particular form of the hull functions.

4.1 Ergodicity

Proposition 4.1 (Particular form of the solution of (2.1))**.** *Assume (A1)-(A5) and let* $p > 0$*. Let* $((u_j)_j, (\xi_j)_j)$ *be the solution of (2.1) with* $u_0(y) = \xi_0(y) = py$*. Then* $((u_j)_j, (\xi_j)_j)$ *satisfies*

$$u_j(t,y) = py + u_j(t,0) \quad \textit{and} \quad \xi_j(t,y) = py + \xi_j(t,0). \tag{4.1}$$

Proof of Proposition 4.1. Using that equation (2.1) is invariant by space translations, invariant by addition of constants, and the fact that for all $y, z \in \mathbb{R}$, $j \in \mathbb{Z}$

$$u_0\left(y+z+\frac{j}{n_0}\right)-pz=u_0\left(y+\frac{j}{n_0}\right) \quad \text{and} \quad \xi_0\left(y+z+\frac{j}{n_0}\right)-pz=\xi_0\left(y+\frac{j}{n_0}\right),$$

we deduce, by the comparison principle, that

$$u_j(t,y+z)-pz=u_j(t,y) \quad \text{and} \quad \xi_j(t,y+z)-pz=\xi_j(t,y).$$

Taking $y=0$, we deduce the result.

□

Proposition 4.2 (Ergodicity). *Assume (A1)-(A5), let $((u_j)_j,(\xi_j)_j)$ be a solution of (2.1) with initial data $u_0(y)=\xi_0(y)=py$ for some $p>0$. Then there exists a constant $\lambda \in \mathbb{R}$ such that, for all $(t,y) \in [0;+\infty)\times\mathbb{R}$, $j \in \{1,...,n_0\}$,*

$$|u_j(t,0)-\lambda t| \leq C_1 \quad \textit{and} \quad |\xi_j(t,0)-\lambda t| \leq C_1, \tag{4.2}$$

and

$$|\lambda| \leq K_1, \tag{4.3}$$

with

$$C_1 = 3p + 4\frac{V_{max}}{\alpha}, \tag{4.4}$$

and K_1 defined in Lemma 2.4.

The proof of Proposition 4.2 is done in different steps, it uses the following classical lemma from ergodic theory (see for instance [13]).

Lemma 4.3. *Consider $\Lambda : \mathbb{R}^+ \to \mathbb{R}$ a continuous function which is sub-additive, meaning that: for all $t, s \geq 0$,*

$$\Lambda(t+s) \leq \Lambda(t) + \Lambda(s).$$

Then $\frac{\Lambda(t)}{t}$ has a limit l as $t \to +\infty$ and

$$l = \inf_{t>0} \frac{\Lambda(t)}{t}. \tag{4.5}$$

Proof of Proposition 4.2.

The main idea of the proof is to control the time oscillations. To do this we will use the following continuous functions for all $T>0$,

$$\lambda_+^u(T) = \sup_{j\in\{1,\ldots,n_0\}} \sup_{t\geq 0} \frac{u_j(t+T,0)-u_j(t,0)}{T},$$

$$\lambda_-^u(T) = \inf_{j\in\{1,\ldots,n_0\}} \inf_{t\geq 0} \frac{u_j(t+T,0)-u_j(t,0)}{T},$$

and

$$\lambda_+^\xi(T) = \sup_{j\in\{1,\ldots,n_0\}} \sup_{t\geq 0} \frac{\xi_j(t+T,0)-\xi_j(t,0)}{T},$$

$$\lambda_-^\xi(T) = \inf_{j\in\{1,...,n_0\}} \inf_{t\geq 0} \frac{\xi_j(t+T,0)-\xi_j(t,0)}{T}.$$

We also introduce

$$\lambda_+(T) = \sup(\lambda_+^u(T), \lambda_+^\xi(T)) \quad \text{and} \quad \lambda_-(T) = \inf(\lambda_-^u(T), \lambda_-^\xi(T)).$$

To get the result, it suffices to prove that $\lambda_+(T)$ and $\lambda_-(T)$ have a common limit λ as $T \to +\infty$ such that $|\lambda_\pm - \lambda| \leq \frac{C_1}{T}$. To do this we would like to apply Lemma 4.3. Because of their definitions, we know that $T \mapsto T\lambda_+^u(T)$ and $T \mapsto T\lambda_+^\xi(T)$ are sub-additive, in the same way $T \mapsto -T\lambda_-^u(T)$ and $T \mapsto -T\lambda_-^\xi(T)$ are also sub-additive. Therefore if $\lambda_\pm^u(T)$ and $\lambda_\pm^\xi(T)$ are finite, we will get the convergence, and we will only have to prove that they have the same limit.

Step 1: $\lambda_+(T)$ and $\lambda_-(T)$ converge as T goes to $+\infty$. We want to use Lemma 4.3. Since $T \mapsto T\lambda_+^u$, $T \mapsto T\lambda_+^\xi$, $T \mapsto -T\lambda_-^u$ and $T \mapsto -T\lambda_-^\xi$ are sub-additive, we deduce that $T \mapsto T\lambda_+$ and $T \mapsto -T\lambda_-$ are sub-additive. It just remains to show that λ_+ and λ_- are bounded (to get a finite limit). Using Lemma 2.7, we get that for all $j \in \{1, ..., n_0\}$ and for all $t, T > 0$, we have

$$-K_1 \leq 0 \leq \frac{u_j(t+T,0) - u_j(t,0)}{T} \leq V_{max} \leq K_1 \tag{4.6}$$

and

$$-K_1 \leq -\max(a_j - \alpha)\frac{V_{max}}{\alpha} \leq \frac{\xi_j(t+T,0) - \xi_j(t,0)}{T} \leq \frac{V_{max}}{\alpha} \max_{j\in\{1,...,n_0\}}(a_j) \leq K_1, \tag{4.7}$$

where we have used the equation satisfied by ξ_j and the facts that

$$-a_j\frac{V_{max}}{\alpha} \leq -(a_j - \alpha)\frac{V_{max}}{\alpha} \leq (a_j - \alpha)(u_j(t,x) - \xi_j(t,x)) \leq 0,$$

and

$$0 \leq V_j(u_{j+1}(t,x) - u_j(t,x)) \leq V_{max}.$$

Step 2: Control on the time oscillations We now prove that λ_+ and λ_- have the same limit. More precisely, we will prove that

$$|\lambda_+(T) - \lambda_-(T)| \leq \frac{C_1}{T},$$

with C_1 defined in Proposition 4.2.

By definition of $\lambda_\pm(T)$, for all $\varepsilon > 0$, there exists $\tau^\pm$ and $v^\pm \in \{u_1, ..., u_n, \xi_1, ..., \xi_n\}$ such that

$$\left| \lambda_\pm(T) - \frac{v^\pm(\tau^\pm + T, 0) - v^\pm(\tau^\pm, 0)}{T} \right| \leq \varepsilon$$

Let us set for all $j \in \{1, ..., n_0\}$,

$$\Delta_j^u = u_j(\tau^+, 0) - u_j(\tau^-, 0), \quad \Delta_j^\xi = \xi_j(\tau^+, 0) - \xi_j(\tau^-, 0)$$

and

$$\Delta = \sup_{j\in\{1,...,n_0\}} \sup(\Delta_j^u, \Delta_j^\xi).$$

Using Proposition 4.1, we have

$$u_j(\tau^+, y) = u_j(\tau^-, y) + u_j(\tau^+, 0) - u_j(\tau^-, 0) \leq u_j(\tau^-, y) + \Delta$$

and

$$\xi_j(\tau^+, y) = \xi_j(\tau^-, y) + \xi_j(\tau^+, 0) - \xi_j(\tau^-, 0) \leq \xi_j(\tau^-, y) + \Delta.$$

Using the comparison principle we get

$$u_j(\tau^+ + T, y) \leq u_j(\tau^- + T, y) + \Delta \tag{4.8}$$

and

$$\xi_j(\tau^+ + T, y) \leq \xi_j(\tau^- + T, y) + \Delta. \tag{4.9}$$

Now we would like to estimate Δ, Let us assume that the maximum in Δ is reached for the index $\bar{j}$. We then have for all $j \in \{1, ..., n_0\}$,

$$\begin{aligned}
\Delta \leq \quad & u_{\bar{j}}(\tau^+, 0) - u_{\bar{j}}(\tau^-, 0) + 2\frac{V_{max}}{\alpha} \\
\leq \quad & u_{j+n_0}(\tau^+, 0) - u_{j-n_0}(\tau^-, 0) + 2\frac{V_{max}}{\alpha} \\
\leq \quad & u_j(\tau^+, 1) - u_j(\tau^-, -1) + 2\frac{V_{max}}{\alpha} \\
\leq \quad & u_j(\tau^+, 0) - u_j(\tau^-, 0) + 2\frac{V_{max}}{\alpha} + 2p,
\end{aligned} \tag{4.10}$$

where we have used for the first line Lemma 2.7 (to compare $u_{\bar{j}}$ and $\xi_{\bar{j}}$), for the second line, the fact that $(u_j)_j$ is non-decreasing in j, for the third line the periodicity of the function u_j, and for the last line we have used Proposition 4.1. Similarly we have

$$\Delta \leq \xi_j(\tau^+, 0) - \xi_j(\tau^-, 0) + 2\frac{V_{max}}{\alpha} + 2p.$$

We now inject this results in (4.8) and (4.9), with $y = 0$, to obtain

$$u_j(\tau^+ + T, 0) - u_j(\tau^+, 0) \leq u_j(\tau^- + T, 0) - u_j(\tau^-, 0) + 2\frac{V_{max}}{\alpha} + 2p,$$

and

$$\xi_j(\tau^+ + T, 0) - \xi_j(\tau^+, 0) \leq \xi_j(\tau^- + T, 0) - \xi_j(\tau^-, 0) + 2\frac{V_{max}}{\alpha} + 2p.$$

Using this two results we get that

$$v^+(\tau^+ + T, 0) - v^+(\tau^+, 0) \leq v^-(\tau^- + T, 0) - v^-(\tau^-, 0) + 4\frac{V_{max}}{\alpha} + 3p,$$

where the possible comparison between ξ_j and u_j adds an additional $2V_{max}/\alpha$, and the possible comparison between u_j and u_k adds an additional p. This implies that

$$T\lambda_+(T) \leq T\lambda_-(T) + 2\varepsilon T + C_1.$$

Since this is true for all $\varepsilon > 0$, we get

$$|\lambda_+(T) - \lambda_-(T)| \leq \frac{C_1}{T}. \tag{4.11}$$

Step 3: Conclusion. From the previous step we know that $\lambda_\pm$ have the same limit. Let us denote it by λ, and by Lemma 4.3 we have, for all $T > 0$,

$$\lambda_-(T) \le \lambda \le \lambda_+(T).$$

Using (4.11), we deduce that

$$|\lambda_\pm - \lambda| \le \frac{C_1}{T}. \tag{4.12}$$

□

4.2 Construction of hull functions

We now would like to prove the existence of time-space global solutions of (2.1).

Proposition 4.4. *Let $p > 0$ and assume (A1)-(A5). Then, there exist some constants $((u_j^\infty(0,0))_j, (\xi_j^\infty(0,0))_j)$ and a real number $\lambda \in \mathbb{R}$ such that for all $(\tau, y) \in \mathbb{R}^2$,*

$$\left(\left(u_j^\infty(\tau,y)\right)_j, \left(\xi_j^\infty(\tau,y)\right)_j\right) = \left(\left(py + \lambda\tau + u_j^\infty(0,0)\right)_j, \left(py + \lambda\tau + \xi_j^\infty(0,0)\right)_j\right),$$

is a solution of (2.1). These constants satisfy, for all $j \in \mathbb{Z}$,

$$u_{j+1}^\infty(0,0) \ge u_j^\infty(0,0) \quad \text{and} \quad \xi_{j+1}^\infty(0,0) \ge \xi_j^\infty(0,0). \tag{4.13}$$

The interest of this result is that if we consider for all $z \in \mathbb{R}$,

$$\begin{cases} h_j(z) = z + u_j^\infty(0,0) & \text{if } j \in \{1,...,n_0\} \\ h_{j+n_0}(z) = h_j(z+p), \end{cases} \quad \begin{cases} g_j(z) = z + \xi_j^\infty(0,0) & \text{if } j \in \{1,...,n_0\} \\ g_{j+n_0}(z) = g_j(z+p), \end{cases} \tag{4.14}$$

then we have directly the following result.

Corollary 4.5. *(Existence of hull functions). Assume (A1)-(A5), then there exists $\lambda \in \mathbb{R}$ such that there exist a hull function $((h_j)_j, (g_j)_j)$ defined as in Definition 1.5.*

Proof of Proposition 4.4.

Step 1: construction of a solution. In this step, we use the functions $((u_j)_j, (\xi_j)_j)$ solution of (2.1) with $u_0(y) = \xi_0(y) = py$. For $m \in \mathbb{R}$, we consider

$$u_j^m(t,0) = u_j(t+m,0) - \lambda m \quad \text{and} \quad \xi_j^m(t,0) = \xi_j(t+m,0) - \lambda m.$$

Since the equation is invariant by addition of constants and by time-translations, we deduce that

$$\left(\left(u_j^m(t,y) = py + u_j^m(t,0)\right)_j, \left(\xi_j^m(t,y) = py + u_j^m(t,0)\right)_j\right), \tag{4.15}$$

is a solution of (2.1). Moreover, u_j^m is Lipschitz continuous in time thanks to Lemma 2.7 and as a consequence ξ_j^m is also Lipschitz continuous in time. Therefore we can use Ascoli Theorem to deduce that there is a sub-sequence $((u_j^m)_j, (\xi_j^m)_j)$ converging uniformly on compact sets to a Lipschitz continuous function $((u_j^\infty)_j, (\xi_j^\infty)_j)$ which satisfies, for all $k \in \mathbb{R}$,

$$\begin{cases} u_j^\infty(t+k,0) = u_j^\infty(t,0) + \lambda k \\ u_j^\infty(t,y) = py + u_j^\infty(t,0) \\ u_{j+1}^\infty \ge u_j^\infty \end{cases} \quad \begin{cases} \xi_j^\infty(t+k,0) = \xi_j^\infty(t,0) + \lambda k \\ \xi_j^\infty(t,y) = py + \xi_j^\infty(t,0) \\ \xi_{j+1}^\infty \ge \xi_j^\infty. \end{cases} \tag{4.16}$$

However, since $k \in \mathbb{R}$, we deduce that

$$u_j^\infty(t,0) = u_j^\infty(0,0) + \lambda t \quad \text{and} \quad \xi_j^\infty(t,0) = \xi_j^\infty(0,0) + \lambda t, \tag{4.17}$$

which implies the result.

□

Proof of Theorem 1.7.

Step 1: Uniqueness of λ. Given some $p \in (0,+\infty)$, let us assume that there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ with their corresponding hull functions $((h_j^1)_j, (g_j^1)_j), ((h_j^2)_j, (g_j^2)_j)$. Then we define for $i = 1, 2$ and $j \in \{1, ..., n_0\}$,

$$u_j^i(t,y) = h_j^i(\lambda_i t + py) \quad \text{and} \quad \xi_j^i(t,y) = g_j^i(\lambda_i t + py),$$

solution of (2.1). Let us denote by $C = \max_{j \in \{1,\ldots,n_0\}} \max_{i \in \{1,2\}} (h_j^i(0), g_j^i(0))$, then we have

$$u_j^1(0,y) \le u_j^2(0,y) + 2C \quad \text{and} \quad \xi_j^1(0,y) \le \xi_j^2(0,y) + 2C.$$

Using the comparison principle we get for all $(t,x) \in (0,+\infty) \times \mathbb{R}$,

$$u_j^1(t,y) \le u_j^2(t,y) + 2C \quad \text{and} \quad \xi_j^1(t,y) \le \xi_j^2(t,y) + 2C.$$

Now we set $y = 0$ to deduce that for all $t \in (0,+\infty)$,

$$h_j^1(\lambda_1 t) \le h_j^2(\lambda_2 t) + 2C \quad \text{and} \quad g_j^1(\lambda_1 t) \le g_j^2(\lambda_2 t) + 2C,$$

which implies that

$$\lambda_1 t \le \lambda_2 t + 4C.$$

Because this is true for all $t \in (0,+\infty)$, we deduce that

$$\lambda_1 \le \lambda_2.$$

The reverse inequality is obtained by exchanging $((h_j^1)_j, (g_j^1)_j)$ and $((h_j^2)_j, (g_j^2)_j)$, which proves that $\lambda_1 = \lambda_2$ and therefore the uniqueness of $\lambda = \bar{F}(p)$.

Continuity of the map $p \mapsto \bar{F}(p)$. This proof is similar to the one in [7] so we skip it.

□

Proof of Theorem 1.4 . It suffices to remark that

$$h_j(z) = z + h_j(0) \quad \text{and} \quad g_j(z) = z + g_j(0).$$

with

$$h_j(0) = \frac{pj}{n_0} \quad \text{and} \quad g_j(0) = \frac{pj}{n_0} + \frac{1}{\alpha} V\left(\frac{p}{n_0}\right), \tag{4.18}$$

is a solution of (1.10) with

$$\lambda = \bar{F}(p) = V\left(\frac{p}{n_0}\right).$$

□

5 Qualitative properties of the effective Hamiltonian

Proof of Theorem 1.9.

Step 1: proof of the lower bound . If we have $p \leq 2h_0 n_0$, then we can see that

$$\left((h_j(z))_j, (g_j(z))_j\right) = \left(\left(z + \frac{pj}{n_0}\right)_j, \left(z + \frac{pj}{n_0}\right)_j\right),$$

is a hull function for $\lambda = 0$. In fact we have

$$\lambda = \alpha(g_j(z) - h_j(z)) = 0,$$

and

$$\begin{aligned} \lambda = \quad & (a_j - \alpha)(h_j(z) - g_j(z)) + \frac{a_j}{\alpha} V_j\left(\frac{p}{n_0}\right) \\ & = 0, \end{aligned}$$

where we have used assumption (A4). Now by uniqueness of the effective Hamiltonian we have that $\bar{F}(p) = 0$.

Step 2: proof of the upper bound. A simple computation gives

$$\bar{F}(p) = V_j(h_{j+1}(0) - h_j(0)), \quad \text{for all } j \in \mathbb{Z}$$

However, we also know that $h_{j+1}(0) \geq h_j(0)$ and that $h_{j+n_0}(0) = h_j(0) + p$ this means that there exists a $i \in \{j, ..., j + n_0\}$ such that $h_{i+1}(0) - h_i(0) \geq p/n_0$. Therefore, using the fact that V_i is non-decreasing, we have

$$V_i\left(\frac{p}{n_0}\right) \leq \bar{F}(p) \leq \min_{j \in \{1,...,n_0\}} (||V_j||_\infty). \tag{5.1}$$

Passing to the limit as p goes to $+\infty$, we get the desired result.

Step 3: proof of the monotonicity. Let $p_1, p_2 \in (0, +\infty)$, and let $\lambda_1 = \bar{F}(p_1), \lambda_2 = \bar{F}(p_2)$ be their respective effective Hamiltonians, each associated to the hull functions $((h_j^1)_j, (g_j^1)_j)$ and $((h_j^2)_j, (g_j^2)_j)$. We assume that $p_2 > p_1$. Therefore, we have

$$h_{n_0}^1(0) - h_0^1(0) = p_1 < p_2 = h_{n_0}^2(0) - h_0^2(0).$$

From this, we can deduce that there exists an integer $k \in \{0, ..., n_0 - 1\}$ such that

$$h_{k+1}^1(0) - h_k^1(0) < h_{k+1}^2(0) - h_k^2(0).$$

Now, using the monotonicity of V_k, we get,

$$V_k\left(h_{k+1}^1(0) - h_k^1(0)\right) \leq V_k\left(h_{k+1}^2(0) - h_k^2(0)\right),$$

which implies that $\lambda_1 \leq \lambda_2$. Therefore the function $\bar{F}$ is non decreasing.

□

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