A junction condition by specified homogenization of a discrete model with a local perturbation and application to traffic flow

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Abstract

In this paper, we focus on deriving traffic flow macroscopic models from microscopic models containing a local perturbation. At the microscopic scale, we consider a first order model of the form "follow the leader" i.e. the velocity of each vehicle depends on the distance to the vehicle in front of it. We consider a local perturbation located at the origin that slows down the vehicles. At the macroscopic scale, we obtain an explicit Hamilton-Jacobi equation left and right of the origin and a junction condition at the origin (in the sense of [18]). As it turns out, the macroscopic model is equivalent to a LWR model, with a flux limiting condition at the junction. Finally, we also present qualitative properties concerning the flux limiter at the junction.

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1 Introduction

The goal of this paper is to derive a macroscopic model for traffic flow problems from a microscopic model. The idea is to rescale the microscopic model, which describes the dynamics of each vehicle individually, in order to get a macroscopic model which describes the dynamics of density of vehicles.

The problem of deriving macroscopic models from microscopic ones has already been studied for models of the type following the leader (i.e. the velocity or the acceleration of each vehicle depends only on the distance to the vehicle in front of it). We refer for example to [5, 8, 21, 16] where the authors rescaled the empirical measure and obtained a scalar conservation law (LWR model). Recently, another approach has been introduced in [11] (see also [10, 12, 13]) where the authors work on the primitive of the empirical measure and at the limit they obtain a Hamilton-Jacobi equation which is the primitive of the LWR model.

The originality of our work is that we assume that there is a local perturbation that slows down the vehicles and we want to understand how this local perturbation influences the macroscopic dynamics. If the local perturbation is located around zero, at the macroscopic scale it is natural to get an Hamilton-Jacobi equation with a junction condition at zero and an effective flux limiter, the difficulty being to construct this effective flux limiter.

Recently, the theory of Hamilton-Jacobi equations with junction or more generally on networks has known important developments in particular since the works of Achdou, Camilli, Cutri and

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Tchou [1] and Imbert, Monneau and Zidani [20]. In this direction, we would like to mention the recent work of Imbert and Monneau [18] in which they give a suitable definition of (viscosity) solutions at the junction which allows to prove comparison principle, stability and so on.

In this paper, we will use the ideas developed in [11] in order to pass from microscopic models to macroscopic ones. In particular, we will show that this problem can be seen as an homogenization result. The difficulty here is that, due to the local perturbation, we are not in a periodic setting and so the construction of suitable correctors is more complicated. In particular, we will use the idea developped in [2], [14] and in the lectures of Lions at the "College de France" [23], which consists in constructing correctors on truncated domains.

2 Main results

2.1 General model: first order model with a single perturbation

In this paper, we are interested in a first order microscopic model of the form

$$\dot{U}_j(t) = V\left(U_{j+1}(t) - U_j(t)\right) \cdot \phi\left(U_j(t)\right), \tag{2.1}$$

where U_j denotes the position of the $j-$th vehicle and $\dot{U}_j$ is its velocity. The function $\phi : \mathbb{R} \to [0, 1]$ simulates the presence of a local perturbation around the origin. We denote by r the radius of influence of the perturbation.

The function V is called the optimal velocity function and we make the following assumptions on V and ϕ:

Assumption (A)

- (A1) $V : \mathbb{R} \to \mathbb{R}^+$ is Lipschitz continuous, non-negative.
- (A2) V is non-decreasing on $\mathbb{R}$.
- (A3) There exists a $h_0 \in (0, +\infty)$ such that for all $h \leq h_0$, $V(h) = 0$.
- (A4) There exists $h_{max} \in (h_0, +\infty)$ such that for all $h \geq h_{max}$, $V(h) = V(h_{max}) =: V_{max}$.
- (A5) The function $p \mapsto pV(-1/p)$ is strictly convex on $[-1/h_0, 0)$.
- (A6) The function $\phi : \mathbb{R} \to [0, 1]$ is Lipschitz continuous and $\phi(x) = 1$ for $|x| \geq r$.

Remark 2.1. *Assumptions (A1)-(A2)-(A3)-(A5) are satisfied by several classical optimal velocity functions, we have added assumption (A4) to work with V' with a bounded support. But by modifying slightly the classical optimal velocity functions, we obtain a function that satisfies all the assumptions. For instance, in the case of the Greenshields based models [15](see also [6]):*

$$V(h) = \begin{cases} 0 & \text{for } h \leq h_0, \\ V_{max}\left(1 - \left(\dfrac{h_0}{h}\right)^n\right) & \text{for } h_0 < h \leq h_{max}, \\ V_{max}\left(1 - \left(\dfrac{h_0}{h_{max}}\right)^n\right) & \text{for } h > h_{max}, \end{cases}$$

with $n \in \mathbb{N}\backslash\{0\}$. Another optimal velocity function, based on the Newell model [24](see also [9]), is given by:

$$V(h) = \begin{cases} 0 & \text{for } h \leq h_0, \\ V_{max}\left(1 - \exp\left(-\left(\dfrac{h-h_0}{b}\right)^n\right)\right) & \text{for } h_0 < h \leq h_{max}, \\ V_{max}\left(1 - \exp\left(-\left(\dfrac{h_{max}-h_0}{b}\right)^n\right)\right) & \text{for } h > h_{max}, \end{cases}$$

with $n \in \mathbb{N}\backslash\{0\}$ and $b \in [0, +\infty)$. See Figure 1 for a schematic representation of an optimal velocity function satisfying assumption (A).

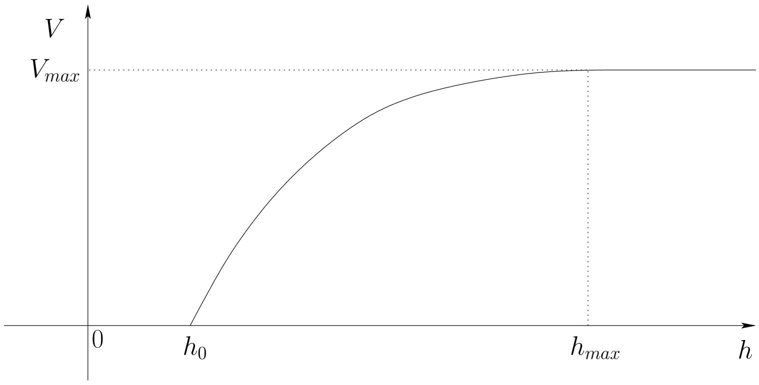

Figure 1: Schematic representation of the optimal velocity function V.

2.2 Injecting the system of ODEs into a single PDE

In this paper, we will study the traffic flow when the number of vehicles per unit length tends to infinity by introducing the rescaled "cumulative distribution function" of vehicles, ρ^ε, defined by

$$\rho^\varepsilon(t,y) = \varepsilon \left(\sum_{i\geq 0} H\left(y - \varepsilon U_i(t/\varepsilon)\right) + \sum_{i<0} \left(-1 + H\left(y - \varepsilon U_i(t/\varepsilon)\right)\right) \right), \tag{2.2}$$

with

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases} \tag{2.3}$$

Under assumption (A), the function ρ^ε is a (discontinuous viscosity) solution (see Theorem 8.1) of the following non-local equation

$$\begin{cases} u_t^\varepsilon + M^\varepsilon \left[\dfrac{u^\varepsilon(t,\cdot)}{\varepsilon}\right](x) \cdot \phi\left(\dfrac{x}{\varepsilon}\right) \cdot |u_x^\varepsilon| = 0 & \text{on } (0,+\infty)\times\mathbb{R} \\ u^\varepsilon(0,x) = u_0(x) & \text{on } \mathbb{R}, \end{cases} \tag{2.4}$$

where M^ε is a non-local operator defined by

$$M^\varepsilon[U](x) = \int_{-\infty}^{+\infty} J(z) E\left(U(x+\varepsilon z) - U(x)\right) dz - \frac{3}{2} V_{max} \tag{2.5}$$

and with

$$E(z) = \begin{cases} 0 & \text{if } z \geq 0 \\ 1/2 & \text{if } -1 \leq z < 0 \\ 3/2 & \text{if } z < -1, \end{cases} \qquad \text{and} \quad J = V' \text{ on } \mathbb{R}. \tag{2.6}$$

Remark 2.2 (Lagrangian formulation). *Another way to treat this problem is to consider a Lagrangian formulation, like in [13], considering the function,*

$$v : [0,T]\times\mathbb{R} \to \mathbb{R}, \quad v(t,y) = U_{\lfloor y \rfloor}(t).$$

This function satisfies for all $(t,y) \in [0,T]\times\mathbb{R}$

$$\begin{cases} v_t(t,y) = V\left(u(t,y+1) - u(t,y)\right) \cdot \phi\left(v(t,y)\right), \\ v(0,y) = v_0(y). \end{cases} \tag{2.7}$$

The difficulty with this formulation is that the function ϕ is evaluated at $v(t,y)$ and not at a physical point of the road. The notion of junction in this case is not well defined and this is why we use the formulation (2.4) *(where the perturbation function is evaluated at a point of the road) instead of* (2.7)*. This will allow us to use the results of Imbert and Monneau [18] concerning quasi-convex Hamiltonians with a junction condition.*

2.3 Convergence result

We define $k_0 = 1/h_0$ and $\overline{H} : \mathbb{R} \to \mathbb{R}$, by

$$\overline{H}(p) = \begin{cases} -p - k_0 & \text{for } p < -k_0, \\ -V\left(\dfrac{-1}{p}\right)|p| & \text{for } -k_0 \le p \le 0, \\ p & \text{for } p > 0. \end{cases} \tag{2.8}$$

Note that such a $\overline{H}$ is continuous, coercive $\left(\lim\limits_{|p|\to+\infty} \overline{H}(p) = +\infty\right)$ and because of (A5), there exists a unique point $p_0 \in [-k_0, 0]$ such that

$$\begin{cases} \overline{H} \text{ is decreasing on } (-\infty, p_0), \\ \overline{H} \text{ is increasing on } (p_0, +\infty). \end{cases} \tag{2.9}$$

We denote by

$$H_0 = \min_{p\in\mathbb{R}} \overline{H}(p) = \overline{H}(p_0) \tag{2.10}$$

and we refer to Figure 2 for an schematic representation of $\overline{H}$.

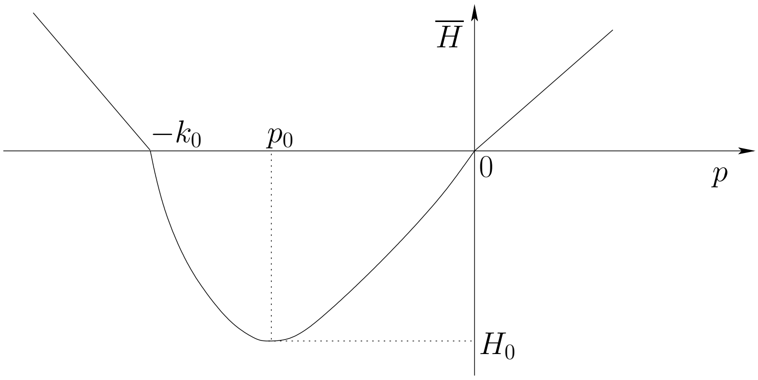

Figure 2: Schematic representation of $\bar{H}$.

The main purpose of this article is to prove that the viscosity solution of (2.4) converges uniformly on compact subsets of $(0,+\infty)\times\mathbb{R}$ as ε goes to 0 to the unique viscosity solution of the following problem

$$\begin{cases} u^0_t + \overline{H}(u^0_x) = 0 & \text{for } (t,x) \in (0,+\infty)\times(-\infty,0) \\ u^0_t + \overline{H}(u^0_x) = 0 & \text{for } (t,x) \in (0,+\infty)\times(0,+\infty) \\ u^0_t + F_{\overline{A}}\left(u^0_x(t,0^-), u^0_x(t,0^+)\right) = 0 & \text{for } (t,x) \in (0,+\infty)\times\{0\} \\ u^0(0,x) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \tag{2.11}$$

where $\overline{A}$ has to be determined and $F_{\overline{A}}$ is defined by

$$F_{\overline{A}}(p_-, p_+) = \max\left(\overline{A}, \overline{H}^+(p_-), \overline{H}^-(p_+)\right), \tag{2.12}$$

with

$$\overline{H}^-(p) = \begin{cases} \overline{H}(p) & \text{if } p \le p_0, \\ \overline{H}(p_0) & \text{if } p \ge p_0, \end{cases} \quad \text{and} \quad \overline{H}^+(p) = \begin{cases} \overline{H}(p_0) & \text{if } p \le p_0, \\ \overline{H}(p) & \text{if } p \ge p_0. \end{cases} \tag{2.13}$$

We also assume that the initial condition satisfies the following assumption:

(A0) (Gradient bound) The function u_0 is Lipschitz continuous and satisfies

$$-k_0 \le (u_0)_x \le 0. \tag{2.14}$$

Remark 2.3. *This condition ensures that initially the vehicles have a security distance between them and since we are working with a first order model, this security distance will be preserved.*

Theorem 2.4 (Junction condition by homogenisation)**.** *Assume (A) and (A0). For $\varepsilon > 0$, let u^ε be the solution of (2.4). Then there exists $\overline{A} \in [H_0, 0]$ such that u^ε converges locally uniformly to the unique viscosity solution u^0 of (2.11) (see Definition 3.4).*

Theorem 2.5 (Junction condition by homogenisation: application to traffic flow)**.** *Assume (A) and that at the initial time, we have, for all $i \in \mathbb{Z}$,*

$$U_i(0) \le U_{i+1}(0) - h_0.$$

We define a function u_0 satisfying (A0) such that for all $\varepsilon > 0$,

$$\rho^\varepsilon(0, x) = \varepsilon \left\lfloor \frac{u_0(x)}{\varepsilon} \right\rfloor. \tag{2.15}$$

Then there exists $\overline{A} \in [H_0, 0]$ such that the function ρ^ε defined by (2.2) converges towards the unique solution u^0 of (2.11)*.*

Remark 2.6. *We notice that in the case of traffic flow,* (2.11) *is equivalent (deriving in space) to a LWR model (see [22, 25]) with a flux limiting condition at the origin. In fact, the fundamental diagram of the model is $pV(1/p)$ and u^0_x corresponds to the density of vehicles.*

The following theorem ensures that when we use (2.11) we only evaluate the function $\overline{H}$ in $[-k_0, 0]$.

Theorem 2.7. *Assume (A0)-(A). Let u^0 be the unique solution of* (2.11)*, then we have for all $(t, x) \in [0, T] \times \mathbb{R}$,*

$$-k_0 \le u^0_x \le 0,$$

with k_0 defined in (A0).

Remark 2.8 (Extension of the effective Hamiltonian)**.** *This theorem implies in particular that in the case of traffic flow, the effective Hamiltonian only needs to be computed for $p \in [-k_0, 0]$. However, for the construction of the correctors it is necessary to work with a coercive Hamiltonian in $\mathbb{R}$ that is why we extend the function $\overline{H}$ in* (2.8)*.*

2.4 Effective Hamiltonian and effective flux-limiter

We define the non-local operator M_p by

$$M_p[U](x) = \int_{-\infty}^{+\infty} J(z) E\left(U(x+z) - U(x) + p \cdot z\right) dz - \frac{3}{2} V_{max}. \tag{2.16}$$

We then have the following result

Proposition 2.9 (Homogenization left and right of the perturbation)**.** *Assume (A). Then for every* $p \in [-k_0, 0]$*, there exists a unique* $\lambda \in \mathbb{R}$*, such that there exists a bounded solution* v *of*

$$\begin{cases} M_p[v](x) \cdot |v_x + p| = \lambda, & x \in \mathbb{R}, \\ v \quad \textit{is } \mathbb{Z}\textit{-periodic}, \end{cases} \tag{2.17}$$

with M_p *defined in* (2.16)*. Moreover, for* $p \in [-k_0, 0]$*, we have* $\lambda = \overline{H}(p)$*.*

To prove this proposition, it is only necessary to notice that $v = 0$ is an obvious solution of (2.17) with $\lambda = \overline{H}(p)$.

To construct the effective flux-limiter $\overline{A}$, we consider the following cell problem: find $\lambda \in \mathbb{R}$ such that there exists a solution w of the following global-in-time Hamilton-Jacobi equation

$$M[w](x) \cdot \phi(x) \cdot |w_x| = \lambda \quad \text{for } x \in \mathbb{R}. \tag{2.18}$$

More precisely, we have the following result.

Theorem 2.10 (Effective flux limiter)**.** *Assume (A). We define the following set of functions*

$$\mathcal{S} = \left\{ w \textit{ s.t. } \exists \textit{ a Lipschitz continuous function } m \textit{ and } C \geq 0 \textit{ s.t. } ||w - m||_{L^\infty(\mathbb{R})} \leq C \right\}.$$

Then we have

$$\overline{A} = \inf \{ \lambda \in \mathbb{R} : \exists\ w \in \mathcal{S} \textit{ solution of (2.18)} \}.$$

2.5 Qualitative properties of the effective flux limiter

Proposition 2.11 (Qualitative properties of the flux limiter)**.** *Assume (A). We have the following qualitative properties on the flux limiter.*

(i) (Monotonicity of the flux-limiter). *Let* $\phi_1, \phi_2 : \mathbb{R} \to [0, 1]$ *be two functions satisfying (A6). Let* $\overline{A}_1$ *and* $\overline{A}_2$ *be their respective flux limiters given by Theorem 2.4. If, for all* $x \in \mathbb{R}$*, we have*

$$\phi_1(x) \leq \phi_2(x), \tag{2.19}$$

then

$$\overline{A}_1 \geq \overline{A}_2. \tag{2.20}$$

(ii) (Flux interruption) *Let* ϕ *be a function satisfying (A6). If* $\phi = 0$ *on an open interval, then we have*

$$\overline{A} = 0. \tag{2.21}$$

2.6 Notations

We recall the definition of the non-local operators that we used in this paper,

$$M[U](x) = \int_{-\infty}^{+\infty} J(z) E\left(U(x+z) - U(x)\right) dz - \frac{3}{2} V_{max}, \tag{2.22}$$

$$M_p[U](x) = \int_{-\infty}^{+\infty} J(z) E\left(U(x+z) - U(x) + p \cdot z\right) dz - \frac{3}{2} V_{max}. \tag{2.23}$$

To each operator M, we associate the operator $\tilde{M}$ which is defined in the same way except that the function E is replaced by the function $\tilde{E}$, defined by

$$\tilde{E}(z) = \begin{cases} 0 \quad \text{if } z > 0 \\ 1/2 \quad \text{if } -1 < z \leq 0 \\ 3/2 \quad \text{if } z \leq -1. \end{cases} \tag{2.24}$$

Remark 2.12. *Using the fact that E and V are bounded, we get that for every function U and every $x \in \mathbb{R}$, we have*

$$-M_0 = -\frac{3}{2}V_{max} \leq M[U](x) \leq 0. \tag{2.25}$$

We also use the following notations for the upper and lower semi-continuous envelopes of a locally bounded function u:

$$u^*(t,x) = \limsup_{s\to t, y\to x} u(s,y) \quad \text{and} \quad u_*(t,x) = \liminf_{s\to t, y\to x} u(s,y).$$

3 Viscosity solutions for (2.4) and (2.11)

3.1 Definitions

In order to give a general definition for all the non-local problems we consider, we will give the definition for the following equation, with $p \in \mathbb{R}$, for all $(t,x) \in (0,+\infty) \times \mathbb{R}$,

$$\begin{cases} u_t + (1-\psi(x)) \cdot M_p[u(t,\cdot)](x) \cdot \phi(x) \cdot |p + u_x| + \psi(x) \cdot \overline{H}(u_x) = 0 \\ u(0,x) = u_0(x), \end{cases} \tag{3.1}$$

with $\psi : \mathbb{R} \to [0,1]$ a Lipschitz continuous function.

Definition 3.1 (Viscosity solutions for (3.1))**.** *Let $T > 0$. An upper semi-continuous function (resp. lower semi-continuous) $u : [0,+\infty)\times\mathbb{R} \to \mathbb{R}$ is a viscosity sub-solution (resp. super-solution) of (3.1) on $[0,T] \times \mathbb{R}$, if $u(0,x) \leq u_0(x)$ (resp. $u(0,x) \geq u_0(x)$) and for all $(t,x) \in (0,T) \times \mathbb{R}$ and for all $\varphi \in C^2([0,T] \times \mathbb{R})$ such that $u - \varphi$ reaches a maximum (resp. a minimum) at the point (t,x), we have*

$$\varphi_t(t,y) + (1-\psi(x)) \cdot \phi(x) \cdot M_p[u(t,\cdot)](x) \cdot |p + \varphi_x(t,x)| + \psi(x)\overline{H}(\varphi_x(t,x)) \leq 0$$

$$\left(\textit{resp.} \quad \varphi_t(t,x) + (1-\psi(x)) \cdot \phi(x) \cdot \tilde{M}_p[u(t,\cdot)](x) \cdot |p + \varphi_x(t,x)| + \psi(x)\overline{H}(\varphi_x(t,x)) \geq 0\right).$$

We say that a function u is a viscosity solution of (3.1) if u^* and u_* are respectively a sub-solution and a super-solution of (3.1).

Remark 3.2. *We use this definition in order to have a stability result for the non-local term. We refer to [7, 26] for such kind of definition and to [11, Proposition 4.2] for the corresponding stability result.*

Definition 3.3 (Class of test functions for (2.11))**.** *We denote by $J_\infty := (0,+\infty) \times \mathbb{R}$, $J_\infty^+ := (0,+\infty) \times (0,+\infty)$ and $J_\infty^- := (0,\infty) \times (-\infty,0)$. We define a class of test functions on Ω by*

$$\mathcal{C}^2(J_\infty) = \left\{\varphi \in C(J_\infty),\ \textit{the restriction of } \varphi \textit{ to } J_\infty^+ \textit{ and to } J_\infty^- \textit{ are } C^2\right\}.$$

Definition 3.4 (Viscosity solutions for (2.11))**.** *Let $\overline{H}$ be given by* (2.8) *and $\overline{A} \in \mathbb{R}$. An upper semi-continuous (resp. lower semi-continuous) function $u : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is a viscosity sub-solution (resp. super-solution) of (2.11) if $u(0, x) \leq u_0(x)$ (resp. $u(0, x) \geq u_0(x)$) and for all $(t, x) \in J_\infty$ and for all $\varphi \in \mathcal{C}^2(J_\infty)$ such that*

$$u \leq \varphi \text{ (resp. } u \geq \varphi\text{)} \quad \text{in a neighbourhood of } (t,x) \in J_\infty \quad \text{and} \quad u(t,x) = \varphi(t,x), \tag{3.2}$$

we have

$$\begin{aligned} &\varphi_t(t,x) + \overline{H}(\varphi_x(t,x)) \leq 0 \quad (\text{resp. } \geq 0) && \text{if } x \neq 0, \\ &\varphi_t(t,x) + F_{\overline{A}}(\varphi_x(t,0^-), \varphi_x(t,0^+)) \leq 0 \quad (\text{resp. } \geq 0) && \text{if } x = 0. \end{aligned} \tag{3.3}$$

We say that a function u is a viscosity solution of (2.11) if u^* and u_* are respectively a sub-solution and a super-solution of (2.11). We refer to this solution as $\overline{A}-$flux limited solution.

3.2 Results for viscosity solutions of (3.1)

Proposition 3.5 (Comparison principle for (3.1))**.** *Assume (A0) and (A). Let u be a sub-solution of (3.1) and v be a super-solution of (3.1). Let us also assume that there exists a constant $K > 0$ such that for all $(t, x) \in [0, T] \times \mathbb{R}$,*

$$u(t,x) \leq u_0(x) + Kt \quad \text{and} \quad -v(t,x) \leq -u_0(x) + Kt. \tag{3.4}$$

Then we have $u(t, x) \leq v(t, x)$ for all $(t, x) \in [0, T] \times \mathbb{R}$.

Proof. The only difficulty in proving the comparison principle comes from the non-local term, but in our case the proof is similar to the proof of [11, Theorem 4.4] and we skip it. □

We now give a comparison principle on bounded sets, to do this, we define for a given point $(t_0, x_0) \in (0, T) \times \mathbb{R}$ and for $\overline{r}, \overline{R} > 0$, the set

$$\mathcal{Q}_{\overline{r},\overline{R}}(t_0, x_0) = (t_0 - \overline{r}, t_0 + \overline{r}) \times (x_0 - \overline{R}, x_0 + \overline{R}).$$

Theorem 3.6 (Comparison principle on bounded sets for (3.1))**.** *Assume (A). Let u be a sub-solution of (3.1) and let v be a super-solution of (3.1) on the open set $\mathcal{Q}_{\overline{r},\overline{R}} \subset (0, T) \times \mathbb{R}$. Also assume that*

$$u \leq v \quad \text{outside } \mathcal{Q}_{\overline{r},\overline{R}},$$

then

$$u \leq v \quad \text{on } \mathcal{Q}_{\overline{r},\overline{R}}.$$

Lemma 3.7 (Existence of barriers for (3.1))**.** *Assume (A0) and (A). There exists a constant $K_1 > 0$ such that*

$$u^+(t,x) = K_1 t + u_0(x) \quad \text{and} \quad u^-(t,x) = u_0(x), \tag{3.5}$$

are respectively super and sub-solutions of (3.1).

Proof. We define $K_1 = M_0 \cdot (|p| + k_0) + |H_0|$. Let us prove that u^+ is a super-solution of (3.1). Using assumption (A0) and the form of the non-local operator and of $\overline{H}$, we have

$$\begin{aligned} \phi(x)(1-\psi(x))M_p[u_0](x) \cdot |p + (u_0)_x| + \psi(x) \cdot \overline{H}((u_0)_x) &\geq -M_0 \cdot |p + (u_0)_x| + H_0 \\ &\geq -M_0(|p| + k_0) - |H_0| = -K_1, \end{aligned} \tag{3.6}$$

where we used (2.25) and (2.10). The proof for u^- is simpler, it uses (2.25) and (2.10),

$$\phi(x)(1-\psi(x))M_p[u_0](x) \cdot |p + (u_0)_x| + \psi(x) \cdot \overline{H}((u_0)_x) \leq 0. \tag{3.7}$$

□

Applying Perron's method (see [19, Proof of Theorem 6], [4] or [17] to see how to apply Perron's method for problems with non-local terms), joint to the comparison principle, we obtain the following result.

Theorem 3.8 (Existence and uniqueness of viscosity solutions for (3.1))**.** *Assume (A0) and (A). Then, there exists a unique solution u of (3.1). Moreover, the function u is continuous and there exists a constant K_1 such that*

$$u_0(x) \leq u(t,x) \leq u_0(x) + K_1 t. \tag{3.8}$$

3.3 Results for viscosity solutions of (2.11)

Now we recall an equivalent definition (Theorem 2.5 in [18]) for sub and super solution at the junction. We will also consider the following problem,

$$u_t + \overline{H}(u_x) = 0 \quad \text{for } t \in (0,T) \text{ and } x \in \mathbb{R}\backslash\{0\}. \tag{3.9}$$

Theorem 3.9 (Equivalent definition for sub/super-solutions)**.** *Let $\overline{H}$ given by* (2.8) *and consider $A \in [H_0, +\infty)$ with H_0 defined in* (2.10)*. Given arbitrary solutions $p^A_\pm \in \mathbb{R}$ of*

$$\overline{H}\left(p^A_+\right) = \overline{H}^+\left(p^A_+\right) = A = \overline{H}^-\left(p^A_-\right) = \overline{H}\left(p^A_-\right), \tag{3.10}$$

let us fix any time independent test function $\phi^0(x)$ satisfying

$$\phi^0_x(0^\pm) = p^A_\pm.$$

Given a function $u : (0,T) \times \mathbb{R} \to \mathbb{R}$, the following properties hold true.

i) If u is an upper semi-continuous sub-solution of (3.9), then u is a H_0-flux limited sub-solution.

ii) Given $A > H_0$ and $t_0 \in (0,T)$, if u is an upper semi-continous sub-solution of (3.9) and if for any test function φ touching u from above at $(t_0,0)$ with

$$\varphi(t,x) = \psi(t) + \phi^0(x), \tag{3.11}$$

for some $\psi \in C^1(0,+\infty)$, we have

$$\varphi_t + F_A(\varphi_x) \leq 0 \quad \text{at } (t_0,0), \tag{3.12}$$

then u is a A-flux limited sub-solution at $(t_0,0)$.

iii) Given $t_0 \in (0,T)$, if u is a lower semi-continuous sub-solution of (3.9) and if for any test function φ satisfying (3.11) touching u from above at $(t_0,0)$ we have

$$\varphi_t + F_A(\varphi_x) \geq 0 \quad \text{at } (t_0,0), \tag{3.13}$$

then u is a A-flux limited super-solution at $(t_0,0)$.

3.4 Control of the oscillations for (2.4)

Theorem 3.10 (Control of the oscillations)**.** *Let $T > 0$. Assume (A0)-(A) and let u be a solution of (2.4), with $\varepsilon = 1$. Then there exists a constants $C_1 > 0$ such that for all $x, y \in \mathbb{R}$, $x \geq y$ and for all $t, s \in [0,T]$, $t \geq s$, we have*

$$0 \leq u(t,x) - u(s,x) \leq C_1(t-s) \quad \text{and} \quad -k_0(x-y) - 1 \leq u(t,x) - u(t,y) \leq 0, \tag{3.14}$$

with k_0 defined in (2.14)*.*

Proof. In this proof we used the barriers given by Lemma 3.7 (with $p = 0$ and $\psi \equiv 0$), which means that the solution u of (2.4) with $\varepsilon = 1$ satisfies for all $(t,x) \in [0,+\infty) \times \mathbb{R}$,

$$0 \leq u(t,x) - u_0(x) \leq M_0 k_0 t. \tag{3.15}$$

In the rest of the proof we will use the following notation:

$$\Omega = \left\{(t,x,y) \in [0,T] \times \mathbb{R}^2 \text{ s.t. } x \geq y\right\}.$$

Proof of the bound on the time derivative. For all $h \geq 0$, we have

$$u(0,x) \leq u(h,x) \leq M_0 k_0 h + u(0,x).$$

Using the fact that equation (2.4) is invariant by addition of constants to the solution and by translations in time, we deduce by the comparison principle that, for all $(t,x) \in [0,+\infty) \times \mathbb{R}$, we have

$$u(t,x) \leq u(t+h,x) \leq M_0 k_0 h + u(t,x).$$

We deduce the result by choosing $C_1 = M_0 k_0$.

Proof of the upper inequality for the control of the space oscillations. We introduce,

$$M = \sup_{(t,x,y)\in\Omega} \left\{ u(t,x) - u(t,y) \right\}.$$

We want to prove that $M \leq 0$. We argue by contradiction and assume that $M > 0$.

Step 1: the test function. For $\eta, \alpha > 0$, small parameters, we define

$$\varphi(t,x,y) = u(t,x) - u(t,y) - \frac{\eta}{T-t} - \alpha x^2 - \alpha y^2.$$

Using (3.15), we have that

$$\varphi(t,x,y) \leq u_0(x) - u_0(y) + 2M_0 k_0 T - \alpha(x^2+y^2) \leq -\alpha(x^2+y^2) + 2M_0 k_0 T, \tag{3.16}$$

where we used assumption (A0) for the second inequality. Therefore we have

$$\lim_{|x|,|y|\to+\infty} \varphi(t,x,y) = -\infty.$$

Since φ is upper-semi continuous, it reaches a maximum at a point that we denote by $(\bar{t},\bar{x},\bar{y}) \in \Omega$. Classically we have for η and α small enough,

$$\begin{cases} 0 < \dfrac{M}{2} \leq \varphi(\bar{t},\bar{x},\bar{y}), \\ \alpha|\bar{x}|, \alpha|\bar{y}| \to 0 \text{ as } \alpha \to 0. \end{cases}$$

Step 2: $\bar{t} > 0$ and $\bar{x} > \bar{y}$. By contradiction, assume first that $\bar{t} = 0$. Then we have

$$\frac{\eta}{T} < u_0(\bar{x}) - u_0(\bar{y}) \leq 0,$$

where we used that u_0 is non-increasing, and we get a contradiction. The fact that $\bar{x} > \bar{y}$, comes directly from the fact that $\varphi(\bar{t},\bar{x},\bar{y}) > 0$.

Step 3: Utilisation of the equation. By doing a duplication of the time variable and passing to the limit we get that

$$\frac{\eta}{(T-\bar{t})^2} \leq \tilde{M}[u(\bar{t},\cdot)](\bar{y}) \cdot |2\alpha\bar{y}| \cdot \phi(\bar{y}) - M[u(t,\cdot)](\bar{x}) \cdot \phi(\bar{x}) \cdot |2\alpha\bar{x}| \leq 2M_0 \cdot \alpha(|\bar{x}| + |\bar{y}|),$$

passing to the limit as α goes to 0, we obtain a contradiction.

Proof of the lower inequality for the control of the space oscillations Let us introduce,

$$M = \sup_{(t,x,y)\in\Omega} \left\{ u(t,y) - u(t,x) - 1 - k_0(x-y) \right\}.$$

We want to prove that $M \leq 0$. We argue by contradiction and assume that $M > 0$.

Step 1: the test function. For $\alpha, \eta > 0$, small parameters we consider the function

$$\varphi(t,x,y) = u(t,y) - u(t,x) - 1 - k_0(x-y) - \alpha|x|^2 - \frac{\eta}{T-t}.$$

We have that

$$\begin{aligned}\varphi(t,x,y) \leq & \quad u_0(y) - u_0(x) - \alpha|x|^2 + 2M_0k_0T - k_0(x-y) - 1 \\ \leq & \quad -\alpha|x|^2 + 2M_0k_0T.\end{aligned}$$

Therefore, we have

$$\lim_{|x|,|y|\to+\infty} \varphi(t,x,y) = -\infty.$$

Using the fact that φ is upper-semi continuous we deduce that φ reaches a maximum at a finite point that we denote $(\bar{t},\bar{x},\bar{y}) \in \Omega$. Classically we have for η and α small enough,

$$\begin{cases} 0 < \dfrac{M}{2} \leq \varphi(\bar{t},\bar{x},\bar{y}), \\ \alpha|\bar{x}| \to 0 \text{ as } \alpha \to 0. \end{cases}$$

Step 2: $\bar{t} > 0$ and $\bar{x} > \bar{y}$. By contradiction, assume that $\bar{t} = 0$. Using the fact that $\varphi(\bar{t},\bar{x},\bar{y}) > 0$ and (A0), we have

$$\frac{\eta}{T} < u(0,\bar{y}) - u(0,\bar{x}) - k_0(\bar{x}-\bar{y}) - 1 \leq -1,$$

which is a contradiction. Hence $\bar{t} > 0$. Using that $\varphi(\bar{t},\bar{x},\bar{y}) > 0$, we also deduce that $\bar{x} > \bar{y}$.

Step 3: Utilisation of the equation By duplicating the time variable and passing to the limit we have that there exists two real numbers a, b, such that $(a, -k_0) \in \overline{\mathcal{D}}^+ u(\bar{t},\bar{y})$, $(b, -k_0) \in \overline{\mathcal{D}}^- u(\bar{t},\bar{x})$ and

$$a - b = \frac{\eta}{(T-\bar{t})^2}. \tag{3.17}$$

Using that u is a sub-solution of (2.4) (with $\varepsilon = 1$), we get

$$a + M[u(\bar{t},\cdot)](\bar{y}) \cdot \phi(\bar{y}) \cdot k_0 \leq 0. \tag{3.18}$$

We claim that

$$M[u(\bar{t},\cdot)](\bar{y}) = \int_{\mathbb{R}} J(z)E(u(\bar{t},\bar{y}+z) - u(\bar{t},\bar{y}))dz - \frac{3}{2}V_{max} = 0.$$

Indeed, let $z \in (h_0, h_{max}]$. If $\bar{y} + z \geq \bar{x}$, using that u is non-increasing in space, we get

$$u(\bar{t},\bar{y}+z) - u(\bar{t},\bar{y}) \leq u(\bar{t},\bar{x}) - u(\bar{t},\bar{y}) \leq -k_0(\bar{x}-\bar{y}) - 1 < -1.$$

If $y + z < x$, using the fact that $\varphi(\bar{t},\bar{x},\bar{y}+z) \leq \varphi(\bar{t},\bar{x},\bar{y})$ we obtain

$$u(\bar{t},\bar{y}+z) - u(\bar{t},\bar{y}) \leq -k_0 z < -1.$$

This implies that we have for all $z \in (h_0, h_{max}]$,

$$E(u(\bar{t},\bar{y}+z) - u(\bar{t},\bar{y})) = \frac{3}{2}.$$

Injecting this in the non-local term, we deduce the claim.

Finally, the fact that $u_t \geq 0$ implies that $a, b \geq 0$. Therefore, inequality (3.18) implies

$$a = 0.$$

Finally, using (3.17), we obtain

$$\frac{\eta}{T^2} \leq 0,$$

which is a contradiction. This ends the proof.

□

4 Correctors for the junction

The key ingredient to prove the convergence result is to construct correctors for the junction. The main result of this section is the existence of appropriate correctors. The proof of this theorem is presented in Section 6. Given $\overline{A} \in \mathbb{R}$, $\overline{A} \geq H_0$, we introduce two real numbers $\overline{p}_+, \overline{p}_- \in \mathbb{R}$, such that

$$\overline{H}\left(\overline{p}_+\right) = \overline{H}^+\left(\overline{p}_+\right) = \overline{H}\left(\overline{p}_-\right) = \overline{H}^-\left(\overline{p}_-\right) = \overline{A}. \tag{4.1}$$

Due to the form of $\overline{H}$ (see (2.8)) this two real numbers exist and are unique.

Theorem 4.1 (Existence of a global corrector for the junction)**.** *Assume (A).*

i) (General properties) There exists a constant $\bar{A} \in [H_0, 0]$ such that there exists a solution w of (2.18) with $\lambda = \overline{A}$ and such that there exists a constant C and a globally Lipschitz continuous function m such that for all $x \in \mathbb{R}$,

$$|w(x) - m(x)| \leq C. \tag{4.2}$$

ii) (Bound from below at infinity) If $\bar{A} > H_0$, then there exists a γ_0 such that for every $\gamma \in (0, \gamma_0)$, we have

$$\begin{cases} w(x+h) - w(x) \geq (\overline{p}_+ - \gamma)h - C & \text{for } x \geq r \text{ and } h \geq 0, \\ w(x-h) - w(x) \geq (-\overline{p}_- - \gamma)h - C & \text{for } x \leq -r \text{ and } h \geq 0. \end{cases} \tag{4.3}$$

iii) (Rescaling w) For $\varepsilon > 0$, we set

$$w^\varepsilon(x) = \varepsilon w\left(\frac{x}{\varepsilon}\right),$$

then (along a subsequence $\varepsilon_n \to 0$) we have that w^ε converges locally uniformly towards a function $W = W(x)$ which satisfies

$$\begin{cases} |W(x) - W(y)| \leq C|x-y| & \text{for all } x, y \in \mathbb{R}, \\ \overline{H}(W_x) = \overline{A} & \text{for all } x \in \mathbb{R}\backslash\{0\}, \end{cases} \tag{4.4}$$

In particular, we have (with $W(0) = 0$)

$$W(x) = \overline{p}_+ x 1_{\{x>0\}} + \overline{p}_- x 1_{\{x<0\}}. \tag{4.5}$$

5 Proof of convergence

This section contains the proof of the main homogenization result (Theorem 2.4). This proof relies on the existences of correctors (Proposition 2.9 and Theorem 4.1).

We begin with two useful lemmas for the proof of Theorem 2.4. The first result is a direct consequence of Perron's method and Lemma 3.7.

Lemma 5.1 (Barriers uniform in ε)**.** *Assume (A0) and (A). There exists a constant $C > 0$ (depending only on M_0 and k_0) such that for all $t > 0$ and $x \in \mathbb{R}$,*

$$|u^\varepsilon(t,x) - u_0(x)| \leq Ct.$$

The following lemma is a direct result of Theorem 3.10.

Lemma 5.2 (Uniform gradient bound)**.** *Assume (A0) and (A). Then the solution u^ε of (2.4) satisfies for all $t > 0$, for all $x, y \in \mathbb{R}$, $x \geq y$,*

$$-k_0(x-y) - \varepsilon \leq u^\varepsilon(t,x) - u^\varepsilon(t,y) \leq 0. \tag{5.1}$$

Before passing to the proof of Theorem 2.4, let us show how it allows us to prove Theorem 2.7.

Proof of Theorem 2.7. We want to prove that for all $t \in [0, +\infty)$ and for all $x, y \in \mathbb{R}$, $x \geq y$,

$$-k_0(x-y) \leq u^0(t,x) - u^0(t,y) \leq 0. \tag{5.2}$$

Using Lemma 5.2, we have that the solution u^ε of (2.4), satisfies for all $(t,x,y) \in [0,+\infty) \times \mathbb{R} \times \mathbb{R}$, with $x \geq y$,

$$-k_0(x-y) - \varepsilon \leq u^\varepsilon(t,x) - u^\varepsilon(t,y) \leq 0. \tag{5.3}$$

Now using Theorem 2.4, passing to the limit as $\varepsilon \to 0$, we obtain the result.

□

We now turn to the proof of Theorem 2.4.

Proof of Theorem 2.4. We introduce

$$\overline{u}(t,x) = \limsup_{\varepsilon \to 0}{}^* u^\varepsilon \quad \text{and} \quad \underline{u}(t,x) = \liminf_{\varepsilon \to 0}{}_* u^\varepsilon. \tag{5.4}$$

Thanks to Lemma 5.1, we know that these functions are well defined. We want to prove that $\overline{u}$ and $\underline{u}$ are respectively a sub-solution and a super-solution of (2.11). In this case, the comparison principle will imply that $\overline{u} \leq \underline{u}$. But, by construction, we have $\underline{u} \leq \overline{u}$, hence we will get $\underline{u} = \overline{u} = u^0$, the unique solution of (2.11).

Let us prove that $\overline{u}$ is a sub-solution of (2.11) (the proof for $\underline{u}$ is similar and we skip it). We argue by contradiction and assume that there exist a test function $\varphi \in \mathcal{C}^2(J_\infty)$ (in the sense of Definition 3.3), and a point $(\bar{t}, \bar{x}) \in (0,+\infty) \times \mathbb{R}$ such that

$$\begin{cases} \overline{u}(\bar{t},\bar{x}) = \varphi(\bar{t},\bar{x}) & & \\ \overline{u} \leq \varphi & \text{on } \mathcal{Q}_{\bar{r},\bar{r}}(\bar{t},\bar{x}) & \text{with } \bar{r} > 0 \\ \overline{u} \leq \varphi - 2\eta & \text{outside } \mathcal{Q}_{\bar{r},\bar{r}}(\bar{t},\bar{x}) & \text{with } \eta > 0 \\ \varphi_t(\bar{t},\bar{x}) + \overline{H}(\bar{x}, \varphi_x(\bar{t},\bar{x})) = \theta & \text{with } \theta > 0, & \end{cases} \tag{5.5}$$

where

$$\overline{H}(\bar{x}, \varphi_x(\bar{t},\bar{x})) := \begin{cases} \overline{H}\left(\varphi_x(\bar{t},\bar{x})\right) & \text{if } \bar{x} \neq 0, \\ F_{\overline{A}}\left(\varphi_x(\bar{t},0^-), \varphi_x(\bar{t},0^+)\right) & \text{if } \bar{x} = 0. \end{cases}$$

Given Lemma 5.2 and (5.4), we can assume (up to changing φ at infinity) that for ε small enough, we have

$$u^\varepsilon \leq \varphi - \eta \quad \text{outside } \mathcal{Q}_{\bar{r},\bar{r}}(\bar{t},\bar{x}).$$

Using the previous lemmas we get that the function $\overline{u}$ satisfies for all $t > 0$ and $x, y \in \mathbb{R}$, $x \geq y$,

$$\begin{gathered} |\overline{u}(t,x) - u_0(x)| \leq Ct, \\ -k_0(x-y) \leq \overline{u}(t,x) - \overline{u}(t,y) \leq 0. \end{gathered} \tag{5.6}$$

First case: $\bar{x} \neq 0$**.** We only consider $\bar{x} > 0$, since the other case ($\bar{x} < 0$) is treated in the same way. We define $p = \varphi_x(\bar{t},\bar{x})$ that according to (5.6) satisfies

$$-k_0 \leq p \leq 0.$$

We choose $\bar{r}$ small enough so that $\bar{x} - 2\bar{r} > 0$. Let us prove that the test function φ satisfies in the viscosity sense, the inequality

$$\varphi_t + \tilde{M}^\varepsilon \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right](x) \cdot \phi\left(\frac{x}{\varepsilon}\right) \cdot |\varphi_x| \geq \frac{\theta}{2} \quad \text{for } (t,x) \in \mathcal{Q}_{\bar{r},\bar{r}}(\bar{t},\bar{x}). \tag{5.7}$$

Let us notice that for ε small enough we have

$$\phi\left(\frac{x}{\varepsilon}\right) = 1 \quad \text{for all } (t,x) \in \mathcal{Q}_{\bar r,\bar r}(\bar t,\bar x).$$

For all $(t,x) \in \mathcal{Q}_{\bar r,\bar r}(\bar t,\bar x)$, we have for $\bar r$ small enough

$$\begin{aligned} \varphi_t(t,x) + \tilde M^\varepsilon\left[\frac{\varphi}{\varepsilon}(t,\cdot)\right](x)\cdot|\varphi_x| &= \varphi_t(\bar t,\bar x) + o_{\bar r}(1) + \tilde M^\varepsilon\left[\frac{\varphi}{\varepsilon}(t,\cdot)\right](x)\cdot|\varphi_x| \\ &= \theta + o_{\bar r}(1) + \tilde M^\varepsilon\left[\frac{\varphi}{\varepsilon}(t,\cdot)\right](x)\cdot|p| - \overline{H}(p) \\ &=: \Delta, \end{aligned} \tag{5.8}$$

where we have used (5.5). We recall that for $-k_0 \le p \le 0$,

$$\overline{H}(p) = M_p[0](0)|p|. \tag{5.9}$$

Moreover, for all $z \in [h_0, h_{max}]$, and for ε and $\bar r$ small enough we have that

$$\begin{aligned} \frac{\varphi(t,x+\varepsilon z) - \varphi(t,x)}{\varepsilon} &= z\varphi_x(t,y) + \varepsilon z^2 \varphi_{xx}(t,\xi(x,x+\varepsilon z)) \\ &\le pz + o_{\bar r}(1) + c\varepsilon, \end{aligned}$$

where we have used the fact that $\varphi \in \mathcal{C}^2$, that $z \in [h_0, h_{max}]$. Now using the fact that $\tilde E$ is decreasing we have

$$\tilde E(pz + c\varepsilon + o_{\bar r}(1)) \le \tilde E\left(\frac{\varphi(t,x+\varepsilon z) - \varphi(t,x)}{\varepsilon}\right).$$

Using this result and replacing the non-local operators in (5.8) by their definition (see 2.16), we obtain

$$\begin{aligned} \Delta \ge \theta + o_{\bar r}(1) + |p| \int_{h_0}^{h_{max}} J(z)\tilde E(pz + c\varepsilon + o_{\bar r}(1))dz \\ - |p| \int_{h_0}^{h_{max}} J(z)\tilde E(pz)dz. \end{aligned} \tag{5.10}$$

We can see that if we have $p = 0$, we obtain directly our result. However, if $-k_0 \le p < 0$,

$$\begin{aligned} \int_{\mathbb{R}} J(z)\tilde E(pz + c\varepsilon + o_{\bar r}(1))dz &= -V\left(\frac{-1 - c\varepsilon + o_{\bar r}(1)}{p}\right) - \frac{1}{2}V\left(-\frac{c\varepsilon + o_{\bar r}(1)}{p}\right) + \frac{3}{2}V_{max}, \\ \int_{\mathbb{R}} J(z)\tilde E(pz)dz &= -V\left(\frac{-1}{p}\right) + \frac{3}{2}V_{max}. \end{aligned} \tag{5.11}$$

Injecting (5.11) in (5.10) and choosing ε and $\bar r$, we obtain

$$\begin{aligned} \Delta \ge\ & \theta + o_{\bar r}(1) + |p|\cdot\left[-V\left(\frac{-1 - c\varepsilon + o_{\bar r}(1)}{p}\right) + V\left(\frac{-1}{p}\right)\right] \\ \ge\ & \theta + o_{\bar r}(1) - ||V'||_\infty \cdot (c\varepsilon + o_{\bar r}(1)) \\ \ge\ & \frac{\theta}{2}, \end{aligned} \tag{5.12}$$

where we have used assumption (A1) for the second line.

Getting a contradiction. By definition, we have for ε small enough,

$$u^\varepsilon \le \varphi - \eta \quad \text{outside } \mathcal{Q}_{\bar r,\bar r}(\bar t,\bar x).$$

Using the comparison principle on bounded subsets for (2.4), we get

$$u^\varepsilon \le \varphi - \eta \quad \text{on } \mathcal{Q}_{\bar r,\bar r}(\bar t,\bar x).$$

Passing to the limit as $\varepsilon \to 0$, we get $\overline{u} \le \varphi - \eta$ on $\mathcal{Q}_{\bar r,\bar r}(\bar t,\bar x)$ and this contradicts the fact that $\overline{u}(\bar t,\bar x) = \varphi(\bar t,\bar x)$.

Second case: $\bar{x} = 0$. Using Theorem 3.9, we may assume that the test function has the following form

$$\varphi(t,x) = g(t) + \overline{p}_- x 1_{\{x<0\}} + \overline{p}_+ x 1_{\{x>0\}} \quad \text{on } \mathcal{Q}_{\bar{r},2\bar{r}}(\bar{t},0), \tag{5.13}$$

where g is a C^1 function defined in $(0,+\infty)$. The last line in condition (5.5) becomes

$$g'(t) + F_{\overline{A}}(\overline{p}_-, \overline{p}_+) = g'(t) + \overline{A} = \theta \quad \text{at } (\bar{t},0). \tag{5.14}$$

Let us consider w the solution of (2.18) provided by Theorem 4.1, and let us denote

$$\varphi^\varepsilon(t,x) = \begin{cases} g(t) + w^\varepsilon(x) & \text{on } \mathcal{Q}_{\bar{r},2\bar{r}}(\bar{t},0), \\ \varphi(t,x) & \text{outside } \mathcal{Q}_{\bar{r},2\bar{r}}(\bar{t},0). \end{cases} \tag{5.15}$$

We would like to prove that this function satisfies in the viscosity sense, for $\bar{r}$ and ε small enough,

$$\varphi^\varepsilon(t,x) + M^\varepsilon\left[\frac{\varphi^\varepsilon}{\varepsilon}(t,\cdot)\right](x)\cdot\phi\left(\frac{x}{\varepsilon}\right)\cdot|\varphi^\varepsilon_x| \geq \frac{\theta}{2} \quad \text{on } \mathcal{Q}_{\bar{r},\bar{r}}(\bar{t},0).$$

Let h be a test function touching φ^ε from below at $(t_1,x_1)\in\mathcal{Q}_{\bar{r},\bar{r}}(\bar{t},0)$, so we have

$$w\left(\frac{x_1}{\varepsilon}\right) = \frac{1}{\varepsilon}\left(h(t_1,x_1) - g(t_1)\right),$$

and

$$w(y) \geq \frac{1}{\varepsilon}\left(h(t_1,\varepsilon y) - g(t_1)\right),$$

for y in a neighbourhood of $\dfrac{x_1}{\varepsilon}$. Since w does not depend on time, we have

$$h_t(t_1,x_1) = g'(t_1).$$

Therefore, we have

$$h_t(t_1,x_1) - g'(t_1) + M[w]\left(\frac{x_1}{\varepsilon}\right)\cdot\phi\left(\frac{x_1}{\varepsilon}\right)\cdot|h_x(t_1,x_1)| \geq \overline{A}.$$

This implies that (using (5.14) and taking $\bar{r}$ small enough)

$$h_t(t_1,x_1) + M[w]\left(\frac{x_1}{\varepsilon}\right)\cdot\phi\left(\frac{x_1}{\varepsilon}\right)\cdot|h_x(t_1,x_1)| \geq \overline{A} + g'(t_1) \geq \frac{\theta}{2},$$

i.e.

$$h_t(t_1,x_1) + M^\varepsilon\left[\frac{\varphi^\varepsilon(t_1,\cdot)}{\varepsilon}\right](x_1)\cdot\phi\left(\frac{x_1}{\varepsilon}\right)\cdot|h_x(t_1,x_1)| \geq \frac{\theta}{2}. \tag{5.16}$$

Getting the contradiction. We have that for ε small enough

$$u^\varepsilon + \eta \leq \varphi = g(t) + \overline{p}_- x 1_{\{x<0\}} + \overline{p}_+ x 1_{\{x>0\}} \quad \text{on } \mathcal{Q}_{\bar{r},2\bar{r}}(\bar{t},0)\backslash\mathcal{Q}_{\bar{r},\bar{r}}(\bar{t},0)$$

Using the fact that $w^\varepsilon \to W$, and using (4.5), we have for ε small enough

$$u^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \quad \text{on } \mathcal{Q}_{\bar{r},2\bar{r}}(\bar{t},0)\backslash\mathcal{Q}_{\bar{r},\bar{r}}(\bar{t},0). \tag{5.17}$$

Combining this with (5.15), we get that

$$u^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \quad \text{outside } \mathcal{Q}_{\bar{r},\bar{r}}(\bar{t},0),$$

By the comparison principle on bounded subsets the previous inequality holds in $\mathcal{Q}_{\bar{r},\bar{r}}(\bar{t},0)$. Passing to the limit as $\varepsilon \to 0$ and evaluating the inequality in $(\bar{t},0)$, we obtain

$$\overline{u}(\bar{t},0) + \frac{\eta}{2} \leq \varphi(\bar{t},0) = \overline{u}(\bar{t},0), \tag{5.18}$$

which is a contradiction.

□

6 Truncated cell problems

This section contains the proof of Theorem 4.1. To do this, we will construct correctors on truncated domains and then pass to the limit as the size of the domain goes to infinity. This idea comes from [2] and [14]. For $l \in (r, +\infty)$, $r << l$ and $r \leq R << l$, we want to find $\lambda_{l,R}$, such that there exists a solution $w^{l,R}$ of

$$\begin{cases} G_R\left(x, [w^{l,R}], w_x^{l,R}\right) = \lambda_{l,R} & \text{if } x \in (-l, l) \\ \overline{H}^-(w_x^{l,R}) = \lambda_{l,R} & \text{if } x \in \{-l\} \\ \overline{H}^+(w_x^{l,R}) = \lambda_{l,R} & \text{if } \in \{l\}, \end{cases} \tag{6.1}$$

with

$$G_R(x, [U], q) = \psi_R(x)\phi(x) \cdot M[U](x) \cdot |q| + (1 - \psi_R(x)) \cdot \overline{H}(q), \tag{6.2}$$

and $\psi_R \in C^\infty$, $\psi_R : \mathbb{R} \to [0, 1]$, with

$$\psi_R \equiv \begin{cases} 1 & \text{on } [-R, R] \\ 0 & \text{outside } [-R-1, R+1], \end{cases} \quad \text{and} \quad \psi_R(x) < 1 \ \forall x \notin [-R, R]. \tag{6.3}$$

To G_R, we associate $\tilde{G}_R$ which is defined in the same way but the operator M is replaced by $\tilde{M}$.

Remark 6.1. *The operator G_R is used to have a local operator near the boundary and then to well define the boundary conditions.*

6.1 Comparison principle for a truncated problem

Proposition 6.2 (Comparison principle on truncated domains)**.** *Let us consider the following problem for $r < l_1 < l_2$ and $\lambda \in \mathbb{R}$, with and $l_2 >> R$.*

$$\begin{cases} G_R(x, [v], v_x) \geq \lambda & \text{for } x \in (l_1, l_2) \\ \overline{H}^+(v_x) \geq \lambda & \text{for } x \in \{l_2\}, \end{cases} \tag{6.4}$$

and for $\varepsilon_0 > 0$

$$\begin{cases} G_R(x, [u], u_x) \leq \lambda - \varepsilon_0 & \text{for } x \in (l_1, l_2) \\ \overline{H}^+(u_x) \leq \lambda - \varepsilon_0 & \text{for } x \in \{l_2\}, \end{cases} \tag{6.5}$$

Then if $u(l_1) \leq v(l_1)$ we have $u \leq v$ in $[l_1, l_2]$.

Proof. The only difficulty in proving this result is the comparison at the boundary $\{l_2\}$. However, for x close to l_2, the function G_R is actually the effective Hamiltonian $\overline{H}$. Therefore, we can proceed as in the proof of [14, Proposition 4.1] and so we skip the proof. □

Remark 6.3. *We have a similar result for $l_1 < l_2 < -r$ and $l_2 << -R$, if the Dirichlet condition is placed at $x = l_2$ and the following conditions are imposed at $x = l_1$:*

$$\begin{cases} \overline{H}^-(v_x) \geq \lambda & \text{for } x \in \{l_1\}, \\ \overline{H}^-(u_x) \leq \lambda - \varepsilon_0 & \text{for } x \in \{l_1\}. \end{cases}$$

6.2 Existence of correctors on a truncated domain

Proposition 6.4 (Existence of correctors on truncated domains)**.** *There exists a unique $\lambda_{l,R} \in \mathbb{R}$ such that there exists a solutions $w^{l,R}$ of (6.1). Moreover, there exists a constant C (depending only on k_0), and a Lipschitz continuous function $m^{l,R}$, such that*

$$\begin{cases} H_0 \leq \lambda_{l,R} \leq 0, \\ |m^{l,R}(x) - m^{l,R}(y)| \leq C|x - y| & \text{for } x, y \in [-l, l], \\ |w^{l,R}(x) - m^{l,R}(x)| \leq C & \text{for } x \in \mathbb{R} \times [-l, l], \end{cases} \tag{6.6}$$

with $H_0 = \min \overline{H}$.

Proof. Given that G_R does not depend explicitly on the time variable, we will classically consider the approximated problem

$$\begin{cases} \delta v^\delta + \psi_R(x) M[v^\delta](x)\cdot\phi(x)\cdot|v^\delta_x| + (1-\psi_R(x))\overline{H}(v^\delta_x) = 0 & \text{for } x\in(-l,l) \\ \delta v^\delta + \overline{H}^-(v^\delta_x) = 0 & \text{for } x\in\{-l\} \\ \delta v^\delta + \overline{H}^+(v^\delta_x) = 0 & \text{for } x\in\{l\} \end{cases} \tag{6.7}$$

Step 1: construction of barriers. Using that 0 and $\delta^{-1}C_0$ are respectively sub and super-solution of (6.7) with $C_0 = |H_0|$, and that we have a comparison principle, we deduce that there exists a continuous viscosity solution, v^δ of (6.7) which satisfies

$$0 \le v^\delta \le \frac{C_0}{\delta}. \tag{6.8}$$

Step 2: control of the space oscillations of v^δ.

Lemma 6.5. *The function v^δ satisfies for all $x, y \in [-l, l]$, $x \ge y$,*

$$-k_0(x-y) - 1 \le v^\delta(x) - v^\delta(y) \le 0, \tag{6.9}$$

with k_0 defined in (A0).

Proof of Lemma 6.5. In the rest of the proof we will use the following notation,

$$\Omega = \left\{(x,y)\in[-l,l]^2 \text{ such that } x\ge y\right\}.$$

Step 2.1: proof of the upper inequality. We want to prove that

$$M = \sup_{(x,y)\in\Omega}\left\{v^\delta(x) - v^\delta(y)\right\} \le 0.$$

We argue by contradiction and assume that $M > 0$. We can see that M is reached for a finite point that we denote by $(\bar{x},\bar{y})\in\Omega$. Given that $M>0$, we deduce that $\bar{x}\neq\bar{y}$. Therefore, we can use the viscosity inequalities for (6.7).

-If $(\bar{x},\bar{y})\in(-l,l)$, we have

$$\begin{aligned} \delta v^\delta(\bar{x}) + G_R(\bar{x},[v^\delta],0) \le 0 \\ \delta v^\delta(\bar{y}) + G_R(\bar{y},[v^\delta],0) \ge 0, \end{aligned}$$

combining these two inequalities with the fact that $G_R(x,[U],0) = 0$, we obtain

$$\delta M \le 0.$$

-If $\bar{x} = l$ and $\bar{y}\in(-l,l)$, similarly we obtain

$$\delta M \le 0,$$

where we have used the fact that $\overline{H}^+(0) = 0$.

-If $\bar{x}\in(-l,l)$ and $\bar{y} = -l$, we obtain

$$\delta M \le H_0 \le 0, \tag{6.10}$$

where we used the fact that $\overline{H}^-(0) = H_0$.

-If $\bar{x} = l$ and $\bar{y} = -l$, we obtain

$$\delta M \le H_0 \le 0. \tag{6.11}$$

For every value of $\bar{x}$ and $\bar{y}$ we obtain a contradiction, therefore we have $M\le 0$.

Step 2.2: proof of the lower inequality. We want to prove that

$$M = \sup_{(x,y)\in\Omega} \left\{ v^\delta(y) - v^\delta(x) - k_0(x-y) - 1 \right\} \leq 0. \tag{6.12}$$

We argue by contradiction and assume that $M > 0$. We can see that M is reached for a finite point that we denote by $(\bar{x}, \bar{y})$. Since $M > 0$, we deduce that $\bar{x} \neq \bar{y}$. Therefore, we can use the viscosity inequalities for (6.7).

Case 1: $\bar{y} \in (-l, l)$**.** If $\bar{y} \in (-l, l)$, we have

$$\delta v^\delta(\bar{y}) + \psi_R(\bar{y}) M[v^\delta](\bar{y}) \cdot \phi(\bar{y}) \cdot |-k_0| + (1 - \psi_R(\bar{y}))\overline{H}(-k_0) \leq 0. \tag{6.13}$$

We claim that $M[v^\delta](\bar{y}) = 0$.

Indeed, for all $z > h_0$, if $\bar{x} > \bar{y} + z$ using the fact that the maximum is reached for $(\bar{x}, \bar{y})$, we deduce that

$$v^\delta(\bar{y} + z) - v^\delta(\bar{x}) - k_0(\bar{x} - \bar{y} - z) - 1 \leq v^\delta(\bar{y}) - v^\delta(\bar{x}) - k_0(\bar{x} - \bar{y}) - 1$$

which implies that

$$v^\delta(\bar{y} + z) - v^\delta(\bar{y}) \leq -k_0 z < -1.$$

On the contrary, if $\bar{x} \leq \bar{y} + z$, using the fact that v^δ is non-increasing in space, we have

$$v^\delta(\bar{y} + z) - v^\delta(\bar{y}) \leq v^\delta(\bar{x}) - v^\delta(\bar{y}) \leq -k_0(\bar{x} - \bar{y}) - 1 < -1.$$

We can therefore, conclude that for all $z \in (h_0, +\infty)$, $E(v^\delta(\bar{y} + z) - v^\delta(\bar{y})) = -\frac{3}{2}$ and so we get $M[v^\delta](\bar{y}) = 0$. Using also that $\overline{H}(-k_0) = 0$, equation (6.13) becomes

$$\delta v^\delta(\bar{y}) \leq 0. \tag{6.14}$$

However, using the fact that $v^\delta \geq 0$ (see (6.8)), we get

$$\delta M \leq \delta v^\delta(\bar{y}) - \delta v^\delta(\bar{x}) \leq 0, \tag{6.15}$$

which is a contradiction.

Case 2: $\bar{y} = -l$**.** In this situation, the viscosity inequality becomes

$$\delta v^\delta(\bar{y}) + \overline{H}^-(-k_0) \leq 0.$$

Using the fact that $\overline{H}^-(-k_0) = \overline{H}(-k_0) = 0$, we obtain

$$\delta v^\delta(\bar{y}) \leq 0,$$

and as in the previous case, we obtain a contradiction. This ends the proof of the lemma.

□

Step 3: construction of a Lipschitz estimate.

Lemma 6.6. *There exists a Lipschitz continuous function m^δ, such that there exists a constant C, (independent of l, R and δ) such that*

$$\begin{cases} |m^\delta(x) - m^\delta(y)| \leq C|x - y| & \text{for all } x, y \in [-l, l], \\ |v^\delta(x) - m^\delta(x)| \leq C & \text{for all } x \in [-l, l]. \end{cases} \tag{6.16}$$

Proof of Lemma 6.6. Let us define m^δ as an affine function in each interval of the form $[ih_0,(i+1)h_0]$, with $i\in\mathbb{Z}$, such that

$$m^\delta(ih_0)=v^\delta(ih_0)\quad\text{and}\quad m^\delta((i+1)h_0)=v^\delta((i+1)h_0).$$

Since m^δ, v^δ are non-increasing and $|v^\delta((i+1)h_0)-v^\delta(ih_0)|\le k_0h_0+1=2$, we deduce that $\forall x\in[ih_0,(i+1)h_0]$,

$$-2\le v^\delta((i+1)h_0)-m^\delta(ih_0)\le v^\delta(x)-m^\delta(x)\le v^\delta(ih_0)-m^\delta((i+1)h_0)\le 2,$$

and for all $x,y\in[-l,l]$,

$$|m^\delta(x)-m^\delta(y)|\le 2k_0|x-y|.$$

□

Step 4: passing to the limit as δ goes to 0. Using (6.8) and (6.16), we deduce that there exists $\delta_n\to 0$ such that

$$\begin{array}{ll} \delta_n v^{\delta_n}(0)\to-\lambda_{l,R} & \text{as } n\to+\infty,\\ m^{\delta_n}-m^{\delta_n}(0)\to m^{l,R} & \text{as } n\to+\infty, \end{array}\tag{6.17}$$

the second convergence being locally uniform. Let us consider,

$$\overline{w}^{l,R}(t,x)=\limsup_{\delta_n\to0}{}^*(v^{\delta_n}-v^{\delta_n}(0))\quad\text{and}\quad \underline{w}^{l,R}=\liminf_{\delta_n\to0}{}_*(v^{\delta_n}-v^{\delta_n}(0)).$$

Therefore, we have that $\lambda_{l,R}, m^{l,R}, \overline{w}^{l,R}$ and $\underline{w}^{l,R}$ satisfy

$$\begin{array}{r} H_0\le\lambda_{l,R}\le 0,\\ |\overline{w}^{l,R}-m^{l,R}|\le C,\\ |\underline{w}^{l,R}-m^{l,R}|\le C,\\ |m_x^{l,R}|\le C. \end{array}\tag{6.18}$$

By stability of the solutions we have that $\overline{w}^{l,R}-2C$ and $\underline{w}^{l,R}$ are respectively a sub-solution and a super-solution of (6.1) and

$$\overline{w}^{l,R}-2C\le\underline{w}^{l,R}.$$

By Perron's method we can construct a solution $w^{l,R}$ of (6.1) and thanks to (6.8) and (6.18), $m^{l,R}$, $\lambda_{l,R}$ and $w^{l,R}$ satisfy (6.6).

The uniqueness of $\lambda_{l,R}$ is classical so we skip it. This ends the proof of Proposition 6.4.

□

Proposition 6.7 (First definition of the flux limiter)**.** *The following limits exist (up to a subsequence)*

$$\begin{cases} \overline{A}_R=\lim\limits_{l\to+\infty}\lambda_{l,R}\\ \\ \overline{A}=\lim\limits_{R\to+\infty}\overline{A}_R. \end{cases}\tag{6.19}$$

Moreover, we have

$$H_0\le\overline{A}_R,\overline{A}\le 0.$$

Proof. This results comes from the fact that we have the following bound on $\lambda_{l,R}$ which is independent of l and R (see Proposition 6.4),

$$H_0 \leq \lambda_{l,R} \leq 0.$$

□

Remark 6.8. *This proposition does not ensure the uniqueness of the flux limiter $\overline{A}$. However, since we know that such a limit exists, we can obtain the converge result. The uniqueness of $\overline{A}$ is given in Theorem 2.10.*

Proposition 6.9 (Control of the slopes on a truncated domain)**.** *Assume that l and R are big enough. Let $w^{l,R}$ be the solution of (6.1) given by Proposition 6.4. We also assume that up to a sub-sequence $\overline{A} = \lim_{R\to+\infty} \lim_{l\to+\infty} \lambda_{l,R} > H_0$. Then there exits a $\gamma_0 > 0$ such that for all $\gamma \in (0, \gamma_0)$, there exists a constant C (independent of l and R) such that for all $x \geq r$ and $h \geq 0$*

$$w^{l,R}(x+h) - w^{l,R}(x) \geq (\overline{p}_+ - \gamma)h - C. \tag{6.20}$$

Similarly, for all $x \leq -r$ and $h \geq 0$,

$$w^{l,R}(x-h) - w^{l,R}(x) \geq (-\overline{p}_- - \gamma)h - C. \tag{6.21}$$

Proof. We only prove (6.20) since the proof for (6.21) is similar. For $\mu > 0$ small enough, we denote by p^μ_+ the real number such that

$$\overline{H}(p^\mu_+) = \overline{H}^+(p^\mu_+) = \lambda_{l,R} - \mu.$$

Using that

$$H_0 < \lambda_{l,R} \leq 0, \tag{6.22}$$

we deduce that p^μ_+ exists, is unique and satisfies $-k_0 \leq p^\mu_+ \leq 0$ for μ small enough.

Let us now consider the function $w^+ = p^\mu_+ x$ that satisfies

$$\overline{H}(w^+_x) = \lambda_{l,R} - \mu \quad \text{for } x \in \mathbb{R}. \tag{6.23}$$

We also have

$$\begin{aligned} M[w^+](x) &= \int_{\mathbb{R}} J(z)E(p^\mu_+(x+z) - p^\mu_+ x)dz - \frac{3}{2}V_{max} \\ &= \int_0^{\frac{-1}{p^\mu_+}} \frac{1}{2}J(z)dz + \int_{\frac{-1}{p^\mu_+}}^{+\infty} \frac{3}{2}J(z)dz - \frac{3}{2}V_{max} \\ &= -V\left(\frac{-1}{p^\mu_+}\right). \end{aligned}$$

For all $x \in (r, l)$, using that $\phi(x) = 1$, we deduce that

$$M[w^+](x) \cdot \phi(x) \cdot |w^+_x| = -V\left(\frac{-1}{p^\mu_+}\right) \cdot |p^\mu_+| = \overline{H}(p^\mu_+) = \lambda_{l,R} - \mu, \tag{6.24}$$

and so the restriction of w^+ to $(r, l]$ satisfies

$$\begin{cases} G_R(x, [w^+], w^+_x) = \lambda_{l,R} - \mu & \text{for } x \in (r, l) \\ \overline{H}^+(w^+_x) = \lambda_{l,R} - \mu & \text{for } x \in \{l\}. \end{cases} \tag{6.25}$$

Let us denote by $g = w^{l,R} - w^{l,R}(x_0)$ and $u = w^+ - w^+(x_0) - 2C$, for some $x_0 \in (r,l)$ and C defined as in Proposition 6.4. Then we have

$$g(x_0) = 0 \geq -2C = u(x_0).$$

Using that g is a solution of (6.4) (with $\varepsilon_0 = \mu$) and u is a solution of (6.5) joint to the comparison principle (Proposition 6.2) we get that

$$w^{l,R}(x) - w^{l,R}(x_0) = g(x) \geq u(x) = p_+^\mu(x - x_0) - 2C.$$

This implies that for all $h \geq 0$ and for all $x \in (r,l)$,

$$w^{l,R}(x+h) - w^{l,R}(x) \geq p_+^\mu h - 2C.$$

Finally, if we choose $\gamma_0 < |p_0 - \bar{p}_+|$ (with p_0 defined in (2.10)), then

$$\overline{H}(\bar{p}_+ - \gamma) = \overline{H}^+(\bar{p}_+ - \gamma),$$

and we can choose $\mu > 0$ such that

$$p_+^\mu = \overline{p}_+ - \gamma.$$

This implies inequality (6.20).

□

Proof of Theorem 4.1. The proof is performed two steps.

Step 1: proof of i) and ii). The goal is to pass to the limit as $l \to +\infty$ and then as $R \to +\infty$. Using Proposition 6.4, there exists $l_n \to +\infty$, such that

$$m^{l_n,R} - m^{l_n,R}(0) \to m^R \quad \text{as } n \to +\infty, \tag{6.26}$$

the convergence being locally uniform. We also define

$$\begin{aligned} \overline{w}^R(x) &= \limsup_{l_n \to +\infty}{}^* \left(w^{l_n,R} - w^{l_n,R}(0)\right), \\ \underline{w}^R(x) &= \liminf_{l_n \to +\infty}{}_* \left(w^{l_n,R} - w^{l_n,R}(0)\right). \end{aligned}$$

Thanks to (6.6), we know that $\overline{w}^R$ and $\underline{w}^R$ are finite and satisfy

$$m^R - C \leq \underline{w}^R \leq \overline{w}^R \leq m^R + C. \tag{6.27}$$

By stability of viscosity solutions, $\overline{w}^R - 2C$ and $\underline{w}^R$ are respectively a sub and a super-solution of

$$G_R(x, [w^R], w_x^R) = \overline{A}_R \quad \text{for } x \in \mathbb{R} \tag{6.28}$$

Therefore, using Perron's method, we can construct a solution w^R of (6.28) with $m^R, \overline{A}^R$ and w^R satisfying

$$\begin{cases} |m^R(x) - m^R(y)| \leq C|x-y| & \text{for all } x, y \in \mathbb{R}, \\ |w^R(x) - m^R(x)| \leq C & \text{for } x \in \mathbb{R} \times \mathbb{R}, \\ H_0 \leq \overline{A}_R \leq 0. \end{cases} \tag{6.29}$$

Using Proposition 6.9, if $\overline{A} > H_0$, we know that there exists a γ_0 and a constant C, such that for all $\gamma \in (0, \gamma_0)$,

$$\begin{cases} w^R(x+h) - w^R(x) \geq (\overline{p}_+ - \gamma)h - C & \text{for all } x \geq r,\ h \geq 0, \\ w^R(x-h) - w^R(x) \geq (-\overline{p}_- - \gamma)h - C & \text{for all } x \leq -r,\ h \geq 0. \end{cases} \tag{6.30}$$

We now pass to the limit as $R \to +\infty$. We consider (up to some subsequence)

$$\begin{cases} \overline{w}(x) = \limsup_{R\to+\infty}{}^* \left(w^R - w^R(0)\right), \\ \underline{w}(x) = \liminf_{R\to+\infty}{}_* \left(w^R - w^R(0)\right), \\ \overline{A} = \lim_{R\to+\infty} \overline{A}_R, \\ m = \lim_{R\to+\infty} (m^R - m^R(0)). \end{cases}$$

The last convergence being locally uniform. Thanks to (6.29), we know that $\overline{w}$ and $\underline{w}$ are finite and satisfy

$$m - C \leq \underline{w} \leq \overline{w} \leq m + C.$$

By stability of viscosity solutions, $\overline{w} - 2C$ and $\underline{w}$ are respectively a sub and a super-solution of (2.18) with $\lambda = \overline{A}$. Using Perron's method, we can then construct a solution w of (2.18) with $\lambda = \overline{A}$ that satisfies (4.2) and (4.3).

Step 2: proof of iii). We are now interested in the rescaled function $w^\varepsilon(x) = \varepsilon w\left(\frac{x}{\varepsilon}\right)$. Using (4.3), we have that

$$w^\varepsilon(x) = \varepsilon m\left(\frac{x}{\varepsilon}\right) + O(\varepsilon).$$

Therefore, we can find a sequence $\varepsilon_n \to 0$, such that

$$w^{\varepsilon_n}(x) \to W(x) \quad \text{locally uniformly as } n \to +\infty, \tag{6.31}$$

with $W(0) = 0$. Like in [18], arguing as in the proof of convergence away from the junction point, we have that W satisfies

$$\overline{H}(W_x) = \overline{A} \quad \text{for } x \neq 0.$$

For all $\gamma \in (0, \gamma_0)$, we have that if $\overline{A} > H_0$ and $x > 0$,

$$W_x \geq \overline{p}_+ - \gamma,$$

where we have used (4.3). Therefore we get

$$W_x = \overline{p}_+ \quad \text{for } x > 0,$$

this result remains valid even if $\overline{A} = H_0$ (in this particular case $W_x = p_0$). Similarly, we get

$$W_x = \overline{p}_- \quad \text{for } x < 0. \tag{6.32}$$

which implies (4.4) and (4.5). This ends the proof of Theorem 4.1.

□

6.3 Proof of Theorem 2.10

Proof of Theorem 2.10. Up to a sub-sequence, we assume that $\overline{A} = \lim_{R\to+\infty} \lim_{l\to+\infty} \lambda_{l,R}$. We want to prove that $\overline{A} = \inf E$, where

$$E = \{\lambda \in [h_0, 0] : \exists\ w \in \mathcal{S} \text{ solution of (2.18)}\},$$

with

$$\mathcal{S} = \{w \text{ s.t. } \exists \text{ a Lipschitz continuous function } m \text{ and a } C > 0 \text{ s.t. } |w(x) - m(x)| \leq C\}.$$

We argue by contradiction and assume that there exists a $\lambda < \overline{A}$ and a function $w^\lambda \in \mathcal{S}$ solution of (2.18). We assume that $w^\lambda(0) = 0$ (if we are not in this situation, we do a translation since we have $w^\lambda - w^\lambda(0) \in \mathcal{S}$). Arguing as in the proof of Theorem 4.1, we deduce that the function

$$w^\varepsilon_\lambda(x) = \varepsilon w^\lambda\left(\frac{x}{\varepsilon}\right)$$

has a limit W^λ (with $W^\lambda(0) = 0$) which satisfies

$$\overline{H}(W^\lambda_x) = \lambda \quad \text{for } x > 0,$$

which means that for all $x > 0$,

$$W^\lambda_x \leq p^\lambda_+ < \overline{p}_+ \quad \text{with } \overline{H}(p^\lambda_+) = \overline{H}^+(p^\lambda_+) = \lambda. \tag{6.33}$$

Similarly we have for all $x < 0$,

$$W^\lambda_x \geq p^\lambda_- > \overline{p}_- \quad \text{with } \overline{H}(p^\lambda_-) = \overline{H}^-(p^\lambda_-) = \lambda. \tag{6.34}$$

These inequalities imply that for all $\gamma > 0$, there exists a constant $\tilde{C}_\gamma > 0$ such that

$$w^\lambda(x) \leq \begin{cases} (p^\lambda_+ + \gamma)x + \tilde{C}_\gamma & \text{for } x > 0, \\ (p^\lambda_- - \gamma)x + \tilde{C}_\gamma & \text{for } x < 0, \end{cases} \tag{6.35}$$

In fact, if w^λ does not satisfies (6.35), we cannot have (6.33) and (6.34). Using Theorem 4.1, we get

$$w^\lambda < w \quad \text{for } |x| \geq \tilde{R}$$

if γ is small enough and $\tilde{R}$ big enough. This implies that there exists a constant $C_{\tilde{R}} > 0$ such that for all $x \in \mathbb{R}$, we have

$$w^\lambda(x) < w(x) + C_{\tilde{R}}.$$

Let us now introduce, $u(t,x) = w(x) + C_{\tilde{R}} - \overline{A}t$ and $u_\lambda(t,x) = w^\lambda(x) - \lambda t$ both solutions of (2.4) with $\varepsilon = 1$ and $u_\lambda(0,x) \leq u(0,x)$. Therefore, the comparison principle implies

$$w^\lambda(x) - \lambda t \leq w(x) + C_{\tilde{R}} - \overline{A}t$$

Dividing by t and passing to the limit as t goes to infinity, we get

$$\overline{A} \leq \lambda,$$

which is a contradiction.

□

7 Qualitative properties of the flux limiter

This section is devoted to the proof of Proposition 2.11.

Proof of Proposition 2.11. We perform the proof of each item separately.

Proof of (i). In order to establish the monotonicity, we have to consider the approximated truncated cell problem (6.7). Let us consider v_1^δ and v_2^δ viscosity solutions of (6.7), respectively for ϕ_1 and ϕ_2, with $0 \le \phi_1 \le \phi_2$. First, using the fact that the non-local operator is negative, we have

$$G_R^2(x,[U],q) \le G_R^1(x,[U],q),$$

with

$$G_R^i(x,[U],q) = \phi_i(x)\cdot M[U](x)\cdot \psi_R(x)\cdot |q| + (1-\psi_R(x))\overline{H}(q), \quad \text{for } i=1,2.$$

Therefore, we have

$$0 = \delta v_1^\delta + G_R^1(x,[v_1^\delta],(v_1^\delta)_x) \ge \delta v_1^\delta + G_R^2(x,[v_1^\delta],(v_1^\delta)_x),$$

meaning that v_1^δ is a sub-solution of (6.7) with ϕ_2. The comparison principle and (6.8) imply that

$$0 \le \delta v_1^\delta \le \delta v_2^\delta \le |H_0|.$$

Passing to the limit as $\delta \to 0$, we obtain

$$0 \ge \lambda_{l,R}^1 \ge \lambda_{l,R}^2 \ge H_0.$$

Passing to the limit as $l, R \to +\infty$, we get the result.

Proof of (ii). If $\phi = 0$ on an open interval, then using [3, Lemme B.1], we can use the definition of a viscosity solution of (6.7) at a point where $\phi = 0$ and therefore, we have

$$\overline{A} = 0.$$

□

8 Link between the system of ODEs and the PDE

This section is devoted to the proof of Theorem 2.5, which is a direct application of our convergence result, Theorem 2.4.

Theorem 8.1. *For $\varepsilon = 1$, the cumulative distribution function ρ defined by* (2.2) *is a discontinuous viscosity solution of*

$$\rho_t + M[\rho(t,\cdot)](x)\cdot\phi(x)\cdot|\rho_x| = 0 \quad \text{for } (t,x)\in[0,+\infty)\times\mathbb{R}. \tag{8.1}$$

Conversely, if u is a bounded and continuous viscosity solution of (8.1) satisfying for some time $T>0$, and for all $t\in(0,T)$

$$u(t,x) \text{ is decreasing in } x,$$

then the points $U_j(t)$, defined by $u(t,U_j(t)) = -(j+1)$ for $j\in\mathbb{Z}$, satisfy the system (2.1) on $(0,T)$.

Before giving the proof of Theorem 8.1, let us do the proof of Theorem 2.5.

Proof of Theorem 2.5. We define a function u_0 satisfying (A0) such that

$$\rho^\varepsilon(0,x) = \rho_0^\varepsilon(x) = \varepsilon\left\lfloor \frac{u_0(x)}{\varepsilon}\right\rfloor.$$

By construction we have

$$(\rho_0^\varepsilon)^*(x) = \rho_0^\varepsilon(x) \le \rho_0^\varepsilon(x) \le u_0(x) < (\rho_0^\varepsilon)_*(x) + \varepsilon. \tag{8.2}$$

Using the fact that ρ^ε is a viscosity solution of (2.4) and the comparison principle (Theorem 3.5) we deduce that (with u^ε the continuous solution of (2.4))

$$\rho^\varepsilon(t,x) \le u^\varepsilon(t,x) \le (\rho^\varepsilon)_*(t,x) + \varepsilon$$

and therefore

$$u^\varepsilon(t,x) - \varepsilon \le \rho^\varepsilon(t,x) \le u^\varepsilon(t,x).$$

Passing to the limit as $\varepsilon \to 0$, we get that $\rho^\varepsilon \to u^0$, which ends the proof of Theorem 2.5.

□

Proof of Theorem 8.1. Theorem 8.1 is a consequence of the following lemma.

Lemma 8.2 (Link between the velocities)**.** *Assume (A). Let $((U_j)_j)$ be the solution of (2.1) with*

$$U_{j+1}(0) - U_j(0) > h_0. \tag{8.3}$$

Then we have

$$\dot{U}_j(t) = -M[u(t,\cdot)](U_j(t)) \cdot \phi(U_j(t)), \tag{8.4}$$

where E and J are defined in (2.6) *and $u(t,x)$ is a continuous function such that*

$$\begin{cases} u(t,x) = \rho_*(t,x) = \rho(t,x) \text{ for } x = U_j(t),\ j \in \mathbb{Z}, \\ u \text{ is decreasing in } x, \end{cases} \tag{8.5}$$

with ρ defined in (2.2) *(with $\varepsilon = 1$).*

Proof. We drop the time dependence to simplify the presentation. Let $j \in \mathbb{Z}$. Using the fact that $u(U_j) = -(j+1)$ and (8.5), we have for all $z \in [0, +\infty)$,

$$\begin{cases} 0 \ge u(U_j + z) - u(U_j) > u(U_{j+1}) - u(U_j) = -1 & \text{if } z \in [0, U_{j+1} - U_j) \\ -1 \ge u(U_j + z) - u(U_j) & \text{if } z \in [U_{j+1} - U_j, +\infty). \end{cases} \tag{8.6}$$

Given that u is continuous, this implies that

$$M[u](U_j) = \int_0^{U_{j+1}-U_j} \frac{1}{2} J(z)dz + \int_{U_{j+1}-U_j}^{+\infty} \frac{3}{2} J(z)dz - \frac{3}{2} V_{max} = -V\left(U_{j+1} - U_j\right). \tag{8.7}$$

Combining this result with (2.1), we obtain (8.4).

□

Noticing that because of (8.5), we have

$$\tilde{M}[\rho_*(t,\cdot)](x) = \tilde{M}[u(t,\cdot)](x) \ge M[u(t,\cdot)](x), \tag{8.8}$$

and using Lemma 8.2, and Definition 3.1, we can see that ρ_* is a discontinuous viscosity super-solution of (8.1). We obtain a similar result for ρ^*, therefore, ρ is a discontinuous viscosity solution of (8.1).

We prove the converse. For the readers convenience we recall Proposition 4.8 from [11] that we will use later. The proof of this proposition remains almost the same in our case the only difference being the definition of the functions E and $\tilde{E}$.

Lemma 8.3. *Assume that $\theta : \mathbb{R} \to \mathbb{R}$ is a non-decreasing and upper semi-continuous (resp. lower semi-continuous). Assume also that*

$$\theta(v) - v \text{ is } 1-\text{periodic in } v.$$

Assume that $\varepsilon = 1$ in (2.4). *Consider also a sub-solution (resp. a super-solution) u of* (2.4). *Then $\theta(u)$ is also a sub-solution (resp. a super-solution) of* (2.4).

Using Lemma 8.3 we can conclude that $\rho_* = \lceil u \rceil$ (resp. $\rho^* = \lfloor u \rfloor$) is a viscosity super-solution (resp. sub-solution) of

$$\partial_t \rho - \tilde{c}(t,x)\partial_x \rho = 0 \quad \text{with } \tilde{c}(t,x) = M[u(t,\cdot)](x) \cdot \phi(x) = \tilde{M}[u(t,\cdot)](x) \cdot \phi(x). \tag{8.9}$$

Using the fact that u is decreasing in space, we define

$$U_i(t) = \inf\{x,\ u(t,x) \leq -(i+1)\} = (u(t,\cdot))^{-1}(-i-1)$$

and we consider the functions $t \mapsto U_i(t)$. They are continuous because u is decreasing in x and is continuous in (t,x).

We now prove that the functions U_i are viscosity solutions of (2.1). Let φ be a test function such that $\varphi(t) \leq U_i(t)$ and $\varphi(t_0) = U_i(t_0)$. Let us now define $\hat{\varphi}(t,x) = -(i+1) + \varphi(t) - x$. It satisfies

$$\hat{\varphi}(t_0, U_i(t_0)) = \rho_*(t_0, U_i(t_0)),$$

and

$$\varphi(\hat{t,x}) \leq \rho_*(t,x) \quad \text{for } U_i(t) - 1 < x < U_{i+1}(t). \tag{8.10}$$

This implies that

$$\begin{aligned} &\varphi_t(t_0) + \tilde{c}(t_0, U_i(t_0)) \geq 0 \\ \Leftrightarrow\ &\varphi_t(t_0) \geq -\tilde{c}(t_0, U_i(t_0)) = -\bar{c}_i(t_0) = V(U_{i+1}(t_0) - U_i(t_0)).\phi(U_i(t_0)). \end{aligned} \tag{8.11}$$

This proves that U_i are viscosity super-solutions of (2.1). The proof for sub-solutions is similar and we skip it. Moreover, since $\bar{c}_i$ is continuous, we deduce that $U_i \in C^1$ and it is therefore a classical solution of (2.1).

□

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References

[1] Y. Achdou, F. Camilli, A. Cutrì, and N. Tchou, *Hamilton-jacobi equations constrained on networks*, Nonlinear Differential Equations and Applications NoDEA, 20 (2013), pp. 413–445.

[2] Y. Achdou and N. Tchou, *Hamilton-jacobi equations on networks as limits of singularly perturbed problems in optimal control: dimension reduction*, <hal-00961015>, (2014).

[3] M. Al Haj, N. Forcadel, and R. Monneau, *Existence and uniqueness of traveling waves for fully overdamped frenkel–kontorova models*, Archive for Rational Mechanics and Analysis, 210 (2013), pp. 45–99.

[4] O. Alvarez and A. Tourin, *Viscosity solutions of nonlinear integro-differential equations*, in Annales de l'Institut Henri Poincaré. Analyse non linéaire, vol. 13, Elsevier, 1996, pp. 293–317.

[5] A. Aw, A. Klar, M. Rascle, and T. Materne, *Derivation of continuum traffic flow models from microscopic follow-the-leader models*, SIAM Journal on Applied Mathematics, 63 (2002), pp. 259–278.

[6] M. Batista and E. Twrdy, *Optimal velocity functions for car-following models*, Journal of Zhejiang University SCIENCE A, 11 (2010), pp. 520–529.

[7] F. Da Lio, C. I. Kim, and D. Slepčev, *Nonlocal front propagation problems in bounded domains with neumann-type boundary conditions and applications*, Asymptotic Analysis, 37 (2004), pp. 257–292.

[8] M. Di Francesco and M. D. Rosini, *Rigorous derivation of the lighthill-whitham-richards˜ model from the follow-the-leader model as many particle limit*, arXiv preprint arXiv:1404.7062, (2014).

[9] L. C. Edie, *Car-following and steady-state theory for noncongested traffic*, Operations Research, 9 (1961), pp. 66–76.

[10] N. Forcadel, C. Imbert, and R. Monneau, *Homogenization of fully overdamped frenkel-kontorova models*, Journal of Differential Equations, 246 (2009), pp. 1057–1097.

[11] ——, *Homogenization of some particle systems with two-body interactions and of the dislocation dynamics*, Discrete and Continuous Dynamical Systems-Series A, 23 (2009).

[12] ——, *Homogenization of accelerated frenkel-kontorova models with n types of particles*, Transactions of the American Mathematical Society, 364 (2012), pp. 6187–6227.

[13] N. Forcadel and W. Salazar, *Homogenization of second order discrete model and application to traffic flow*, <hal-00966729>, (2014).

[14] G. Galise, C. Imbert, and R. Monneau, *A junction condition by specified homogenization*, arXiv preprint arXiv:1406.5283, (2014).

[15] B. Greenshields, W. Channing, H. Miller, et al., *A study of traffic capacity*, in Highway research board proceedings, vol. 1935, National Research Council (USA), Highway Research Board, 1935.

[16] D. Helbing, *From microscopic to macroscopic traffic models*, in A perspective look at nonlinear media, Springer, 1998, pp. 122–139.

[17] C. Imbert, *A non-local regularization of first order hamilton–jacobi equations*, Journal of Differential Equations, 211 (2005), pp. 218–246.

[18] C. Imbert and R. Monneau, *Flux-limited solutions for quasi-convex hamilton-jacobi equations on networks*, <hal-00832545>, (2014).

[19] C. Imbert, R. Monneau, and E. Rouy, *Homogenization of first order equations with (u/ε)-periodic hamiltonians part ii: Application to dislocations dynamics*, Communications in Partial Differential Equations, 33 (2008), pp. 479–516.

[20] C. Imbert, R. Monneau, and H. Zidani, *A hamilton-jacobi approach to junction problems and application to traffic flows*, ESAIM: Control, Optimisation and Calculus of Variations, 19 (2013), pp. 129–166.

[21] H. Lee, H.-W. Lee, and D. Kim, *Macroscopic traffic models from microscopic car-following models*, Physical Review E, 64 (2001), p. 056126.

[22] M. J. Lighthill and G. B. Whitham, *On kinematic waves. ii. a theory of traffic flow on long crowded roads*, Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences, 229 (1955), pp. 317–345.

[23] P. L. Lions, *Lectures at collège de france*, 2013-2014.

[24] G. F. Newell, *Nonlinear effects in the dynamics of car following*, Operations Research, 9 (1961), pp. 209–229.

[25] P. I. Richards, *Shock waves on the highway*, Operations research, 4 (1956), pp. 42–51.

[26] D. Slepčev, *Approximation schemes for propagation of fronts with nonlocal velocities and neumann boundary conditions*, Nonlinear Analysis: Theory, Methods & Applications, 52 (2003), pp. 79–115.