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Abstract

In the present paper we consider a locally perturbed transport stochastic equation. First, we prove an existence and uniqueness result for a global solution. We then prove an homogenization result for this equation.

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1 Introduction

1.1 General model

The goal of this paper is to present a homogenization result for a stochastic Burgers' type equation perturbed by white noise. We consider the following perturbed transport equation with a viscosity term and with a random force which is a white noise in space and time,

$$du^\delta(t,x) = \left(\mu \frac{\partial^2 u^\delta(t,x)}{\partial x^2} + \beta\left(\frac{x}{\delta}\right)\frac{\partial}{\partial x} g(u^\delta(t,x))\right)dt + dW, \quad x\in[0,1],\ t>0. \tag{1.1}$$

Here u represents a certain density in space and time and $g:\mathbb{R}\to[0,+\infty)$ denotes the flux. Finally, μ is a postive constant, the function $\beta:\mathbb{R}\to[0,+\infty)$ is a local perturbation and W is a Q-Wiener process in $L^2(0,1)$.

More precisely, we consider an operator Q which is a trace class non-negative operator on $L^2(0,1)$ and we define

$$W(t) = \sum_{k=1}^{\infty}\sqrt{\lambda_k} e_k \beta_k(t),\ t\geq 0,$$

where $\{e_k\}$ is an orthonormal basis of $L^2(0,1)$, $\{\lambda_k\}$ is the family of eigenvalues of the operator Q (i.e. $Qe_k=\lambda_k e_k,\quad k\in\mathbb{N}$), and $\{\beta_k\}_k$ is a sequence of mutually independent real Brownian motions in a probability space $(\Omega,\mathcal{F},\mathbb{P})$, adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

The model is inspired from the stochastic Burgers' equation which is used to study turbulent flows in the presence of random forces. Note that, even if the deterministic model is not realistic because it does not display any chaos, the situation is different when the force is a random one, as in our case.

Several authors studied the stochastic Burgers equation in one dimension, driven by additive or multiplicative noise as model describing turbulences (see [2], [3], [6], [7]). A result concerning the homogenization was proved for the integro-differential deterministic Burgers equation in [1]. Equation (1.1) is also similar to the LWR model (see for instance [8], [9]) for traffic flow.

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To the best of our knowledge, this is the first paper which treats the stochastic equation (1.1) and its homogenization.

The problem (1.1) shall be studied with the following boundary conditions,

$$u^\delta(0,t) = u^\delta(1,t) = 0, \tag{1.2}$$

and the initial condition,

$$u^\delta(0,x) = u_0(x), \quad \text{for all } x \in [0,1]. \tag{1.3}$$

We reformulate (1.1)-(1.2)-(1.3) as the following abstract stochastic equation,

$$\begin{cases} du^\delta = \left(Au^\delta + B^\delta g(u^\delta)\right) dt + dW, \\ u^\delta(0) = u_0, \end{cases} \tag{1.4}$$

where the self-adjoint operator A on $L^p(0,1)$ is defined by

$$Au = \mu \frac{\partial^2}{\partial x^2} u \quad \text{for } u \in D(A) = \{u \in H^2(0,1) \ s.t. \ u(0) = u(1) = 0\}, \tag{1.5}$$

and the operator B^δ on $L^p(0,1)$, for $p \geq 3$, is defined, for all $\delta > 0$, by

$$B^\delta u = \beta\left(\frac{x}{\delta}\right) \frac{\partial}{\partial x} u. \tag{1.6}$$

In fact, since A is a self-adjoint negative operator in $L^2(0,1)$, we have

$$Ae_k = -\alpha_k e_k, \quad n \in \mathbb{N},$$

where $\{\alpha_k\}$ is the sequence of positive eigenvalues.

We consider the following assumptions

(A1) (Uniform bound on β) We have for all $x \in \mathbb{R}$,

$$0 \leq \beta(x) \leq ||\beta||_\infty < +\infty, \quad \text{and} \quad 0 \leq g(x) \leq ||g||_\infty < +\infty.$$

(A2) (Regularity) We have $\beta \in C^1(\mathbb{R})$ and $||\beta'|| < +\infty$ and g is a Lipschitz continuous function and we denote $||g'||_\infty$ its Lipschitz constant.

(A3) (Asymptotic behaviour) There exits a function $\overline{\beta} \in C^1$ such that

$$\beta\left(\frac{\cdot}{\delta}\right) \to \overline{\beta}, \text{ as } \delta \to 0 \text{ in } W^{1,p}(\mathbb{R}). \tag{1.7}$$

We also assume that $\overline{\beta}$ satisfies

$$0 \leq \overline{\beta}(x) \leq ||\overline{\beta}||_\infty < +\infty \quad \text{and} \quad ||\overline{\beta}'|| < +\infty.$$

(A4) (Regularity of the noise) The eigenvalues of the operators Q and A satisfy

$$\sum_{k=1}^{\infty} \frac{\lambda_k}{\alpha_k^{1-2\alpha}} < +\infty,$$

for some $\alpha \in \left(0, \dfrac{1}{2}\right)$.

In this framework we shall prove existence and uniqueness of a solution to the equation describing the inhomogeneous model (1.4) for each δ fixed, followed by the convergence of this solution, for $\delta \to 0$, to the one of the following homogeneous model.

$$\begin{cases} d\overline{u} = \left(A\overline{u} + \overline{B}g(\overline{u})\right) dt + dW, \\ \overline{u}(0) = u_0, \end{cases} \tag{1.8}$$

where $\overline{B}\overline{u} = \overline{\beta}(x) \dfrac{\partial}{\partial x} \overline{u}$.

1.2 Main results

Our first main result is a result of existence and uniqueness of a solution for (1.4).

Theorem 1.1 (Existence and uniqueness of a mild solution for (1.4))**.** *Assume (A1), (A2), and (A4). Let u_0 be given which is $\mathcal{F}_0-$measurable and such that $u_0 \in L^p(0,1)$ a.s. in Ω, for some $p \geq 3$. Let $T > 0$ and $\delta > 0$. Then there exists a unique mild solution (see Definition 2.1) of equation* (1.4)*, which belongs, a.s. in Ω, to $C([0,T], L^p(0,1))$.*

Remark 1.2. *If $\overline{\beta}$ is defined as in (A3), then Theorem 1.1 implies the existence of a unique mild solution for* (1.8)*.*

Our second main result concerns the homogenization of (1.4).

Theorem 1.3 (Homogenization result for (1.4))**.** *Assume (A1)-(A4). Let $T > 0$. Let $(u^\delta)_\delta$ be the sequence of mild solutions of* (1.4) *in $[0,T]$, provided by Theorem 1.1. Let $\overline{u}$ be the mild solution of* (1.8) *in $[0,T]$. Then*

$$u^\delta \to \overline{u} \text{ as } \delta \to 0 \quad \text{in } C([0,T], L^p(0,1)).$$

The organisation of the article is the following: Section 2 contains the definition of a mild solution and some preliminary results. Section 3 contains the proof of the result of existence and uniqueness of the solution while Section 4 contains the proof of the homogenization result.

2 Definitions and preliminary results

2.1 Definition of the mild solution for (1.4)

Let us begin by introducing the definition of a mild solution. For the readers convenience, we recall that the unique solution of the linear problem

$$\begin{cases} du = Audt + dW, \\ u(0) = 0. \end{cases} \tag{2.1}$$

is given by the stochastic convolution,

$$W_A(t) = \int_0^t e^{(t-s)A} dW(s). \tag{2.2}$$

Keeping in mind that, for a constant $C > 0$, we have for the eigenvalues of A that

$$|e_k(\xi)| \leq C, \quad k \in \mathbb{N}, \quad \xi \in [0,1],$$

and also considering the properties of the semi-group generated by the Laplace operator (see Lemma 2.2 below), we can apply [4, Theorem 5.24] and get that

$$W_A \in C([0,T] \times [0,1]) \cap C([0,T]; W^{1,p}(0,1)),$$

$p \geq 3$. Furthermore, W_A is a $\alpha-$Hölder continuous function for all $\alpha \in (0, 1/4)$ with respect to the variables t and x.

We can now give the definition of the solution.

Definition 2.1. *A stochastic process $u^\delta \in C([0,T]; L^p(0,1))$ is called mild solution of problem (1.4) on $[0,T]$ if for each starting point $u_0 \in L^p(0,1)$, $p \geq 3$, we have that*

$$u^\delta(t) = e^{tA} u_0 + \int_0^t e^{(t-s)A} B^\delta g(u^\delta) ds + W_A \text{ a.s. in } \Omega,$$

for all $t \in [0,T]$.

2.2 Preliminary results

The idea which shall be used for both existence and homogenization results consists in rewriting the stochastic equations as random differential equations.

In fact, for each fixed $\omega \in \Omega$, we shall rewrite equation (1.4) in the form

$$\begin{cases} \dfrac{dv^\delta}{dt} = Av^\delta + B^\delta g(v^\delta + W_A), \\ v^\delta(0) = u_0 \end{cases} \tag{2.3}$$

by using the change of variable

$$v^\delta(t) = u^\delta(t) - W_A(t), \text{ for all } t \geq 0. \tag{2.4}$$

The mild solution corresponding to (2.3) will be then given by

$$v^\delta(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}B^\delta g(v^\delta + W_A)ds. \tag{2.5}$$

Using this form of the equation we shall prove that v^δ (and implicitly u^δ) exists and is unique. Like before, for the homogeneous problem (1.8) we can define $\overline{v} = \overline{u} - W_A$, that satisfies

$$\begin{cases} \dfrac{d\overline{v}}{dt} = A\overline{v} + \overline{B}g(\overline{v} + W_A), \\ \overline{v}(0) = u_0, \end{cases} \tag{2.6}$$

which mild solution is given by

$$\overline{v} = e^{At}u_0 + \int_0^t e^{A(t-s)}u_0\overline{B}g(\overline{v} + W_A)ds. \tag{2.7}$$

The idea used to prove homogenization is then to obtain the convergence result of v^δ to $\overline{v}$, which will imply the same for u^δ and $\overline{u}$.

In order to see the properties of the stochastic convolution, we recall the following result concerning contraction semi-group on $L^p(0,1)$ (see for instance [10, Lemma 3, Part I]).

Lemma 2.2. *For any $s_1 \leq s_2 \in \mathbb{R}$, and $r \geq 1$, $e^{At} : W^{s_1,r}(0,1) \to W^{s_2,r}(0,1)$, for all $t > 0$ and we have that there exists a constant C_1 depending only on s_1, s_2 and r such that*

$$\left|e^{tA}z\right|_{W^{s_2,r}(0,1)} \leq C_1\left(t^{\frac{s_1-s_2}{2}} + 1\right)|z|_{W^{s_1,r}(0,1)}, \text{ for all } z \in W^{s_1,r}(0,1). \tag{2.8}$$

Finally, we present a useful (deterministic) result that we will use several times in the rest of this paper.

Lemma 2.3. *Assume (A1)-(A2) and (A4). There exists two constants $C_1, C_2 > 0$ such that for all $t \geq 0$ and $v \in C([0,T]; L^p(0,1))$ we have*

$$\left|e^{tA}\beta(x)\frac{\partial}{\partial x}g(v)\right|_{L^p(0,1)} \leq C_1C_2\left(t^{-\frac{1}{2}-\frac{1}{2p}} + 1\right)\left(|\beta|_{L^p(0,1)} + |\beta'|_{L^p(0,1)}\right)|g(v)|_{L^p(0,1)}. \tag{2.9}$$

Proof of Lemma 2.3. Using the Sobolev embedding theorem (the reader is referred to the Hitchhiker's guide of Nezza, Palatucci and Valdinoci [5] for results concerning fractional Sobolev spaces), we have that there exists a constant $C_2 > 0$ such that

$$\begin{aligned} \left|e^{tA}\beta(x)\frac{\partial}{\partial x}g(v)\right|_{L^p(0,1)} &\leq C_2\left|e^{tA}Bg(v)\right|_{W^{\frac{1}{p},\frac{p}{2}}(0,1)} \\ &\leq C_1C_2\left(t^{-\frac{1}{2}-\frac{1}{2p}} + 1\right)|Bg(v)|_{W^{-1,\frac{p}{2}}(0,1)}, \end{aligned} \tag{2.10}$$

where we used Lemma 2.2 for the second inequality, with $s_1 = -1,\ s_2 = 1/p$ and $r = p/2$. By definition, we have that

$$|Bg(v)|_{W^{-1,\frac{p}{2}}(0,1)} = \sup\left\{\langle Bg(v), u\rangle /\ u \in W_0^{1,\frac{p}{p-2}}(0,1), |u|_{W_0^{1,\frac{p}{p-2}}(0,1)} \le 1\right\}.$$

However, for all $u \in W_0^{1,\frac{p}{p-2}}(0,1),\ |u|_{W_0^{1,\frac{p}{p-2}}(0,1)} \le 1$, we have

$$\begin{aligned}
|\langle Bg(v+W_A), u\rangle| =& \left|\int_0^1 \beta(x)\frac{\partial}{\partial x}g(v)udx\right| \\
=& \left|\int_0^1 g(v)\left(\beta'(x)u + \beta\frac{\partial u}{\partial x}\right)dx\right| \\
\le& \left|\int_0^1 g(v)\beta'(x)udx\right| + \left|\int_0^1 g(v)\beta\frac{\partial u}{\partial x}dx\right| \\
\le& \left(\int_0^1 |g(v)|^{p/2}|\beta'|^{p/2}dx\right)^{2/p}\cdot\left(\int_0^1 |u|^{p/(p-2)}dx\right)^{(p-2)/p} \\
&+ \left(\int_0^1 |g(v)|^{p/2}|\beta|^{p/2}dx\right)^{2/p}\cdot\left(\int_0^1 \left|\frac{\partial u}{\partial x}\right|^{p/(p-2)}dx\right)^{(p-2)/p} \\
\le& \left(\int_0^1 |g(v)|^{p/2}|\beta'|^{p/2}dx\right)^{2/p} + \left(\int_0^1 |g(v)|^{p/2}|\beta|^{p/2}dx\right)^{2/p} \\
\le& |g(v)|_{L^p(0,1)}\left(|\beta|_{L^p(0,1)} + |\beta'|_{L^p(0,1)}\right),
\end{aligned}$$

where we have used for the fourth and fifth line the Hölder inequality with coefficients $p/2$ and $p/(p-2)$, for the sixth line the fact that $|u|_{W_0^{1,\frac{p}{p-2}}(0,1)} \le 1$ and finally, for the seventh line we used again Hölder inequality with coefficients 2. Injecting this result in (2.10), we obtain (2.9). This concludes the proof of Lemma 2.3.

□

3 Existence and uniqueness of the solution for the inhomogeneous model

This section contains the proof of Theorem 1.1, which is a direct consequence of combining (2.4) with the following lemma.

Lemma 3.1 (Existence and uniqueness of a mild solution for (2.3))**.** *Assume (A1), (A2), and (A4). Let u_0 be given which is $\mathcal{F}_0-$measurable and such that $u_0 \in L^p(0,1)$ a.s. in Ω, for some $p \ge 3$. Let $T > 0$ and $\delta > 0$. Then there exists a unique mild solution of* (2.3)*, which belongs a.s. to $C([0,T], L^p(0,1))$.*

The proof of Lemma 3.1 is done in two parts, first we prove that (2.3) admits a solutions locally in time and then we prove the existence of a solution global in time. In order to simplify the notations, we drop the dependence in δ in the rest of the section.

We recall that (2.3) is s deterministic equation, for $\omega \in \Omega$ fixed.

3.1 Local existence in time

Lemma 3.2 (Local existence in time for (2.3))**.** *For any $p \ge 3$, and $m > |u_0|_{L^p(0,1)}$, there exists a stopping time $T^* > 0$ such that* (2.3) *has a unique mild solution of the form* (2.5) *in*

$$\Sigma_p(m, T^*) = \left\{v \in C([0,T^*], L^p(0,1)) :\ |v(t)|_{L^p(0,1)} \le m\ \forall t \in [0,T^*]\right\}.$$

Proof of Lemma 3.2. This proof is done by a fixed point argument in $\Sigma_p(m,T^*)$. Therefore, we want to prove that for all $v \in \Sigma_p(m,T^*)$, the transformation $Gv = z$ defined by

$$z(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}\beta(x)\frac{\partial}{\partial x}g(v+W_A)ds,$$

is a contraction from $\Sigma_p(m,T^*)$ into $\Sigma_p(m,T^*)$.

We define, for all $v \in \Sigma_p(m,T^*)$, the norm $||\cdot||_{\Sigma_p(m,T^*)}$ by

$$||v||_{\Sigma_p(m,T^*)} = \sup_{t\in[0,T^*]} |v|_{L^p(0,1)}.$$

Step 1: G is stable in $\Sigma_p(m,T^*)$. Let us first prove that $z = Gv$ is in $\Sigma_p(m,T^*)$. First we have,

$$|z(t)|_{L^p(0,1)} \leq \left|e^{tA}u_0\right|_{L^p(0,1)} + \int_0^t \left|e^{(t-s)A}Bg(v+W_A)\right|_{L^p(0,1)} ds. \tag{3.1}$$

Let us now treat the term inside the integral, using Lemma 2.3, we obtain that

$$\begin{aligned}\left|e^{(t-s)A}\beta(x)\frac{\partial}{\partial x}g(v+W_A)\right|_{L^p(0,1)} &\leq C_1C_2\left((t-s)^{-\frac{1}{2}-\frac{1}{2p}}+1\right)\left(|\beta|_{L^p(0,1)}+|\beta'|_{L^p(0,1)}\right)|g|_{L^p(0,1)} \\ &\leq C_1C_2\left((t-s)^{-\frac{1}{2}-\frac{1}{2p}}+1\right)\left(||\beta||_\infty+||\beta'||_\infty\right)||g||_\infty. \end{aligned}\tag{3.2}$$

Since e^{At} is a contraction on $L^p(0,1)$, we have that $|e^{tA}u_0|_{L^p(0,1)} \leq |u_0|_{L^p(0,1)}$.

Injecting this result and (3.2) into (3.1) we obtain

$$\begin{aligned}|z(t)|_{L^p(0,1)} &\leq |u_0|_{L^p(0,1)} + C_1C_2\left(||\beta||_\infty+||\beta'||_\infty\right)||g||_\infty \int_0^t \left((t-s)^{-\frac{1}{2}-\frac{1}{2p}}+1\right)ds \\ &\leq |u_0|_{L^p(0,1)} + C_1C_2\left(||\beta||_\infty+||\beta'||_\infty\right)||g||_\infty \left(\frac{2p}{p-1}{T^*}^{\frac{1}{2}-\frac{1}{2p}}+T^*\right).\end{aligned}$$

We can see that if $m > |u_0|_{L^p(0,1)}$, given that $p \geq 3$, there exists a time T^* such that

$$|u_0|_{L^p(0,1)} + C_1C_2\left(||\beta||_\infty+||\beta'||_\infty\right)||g||_\infty \left(\frac{2p}{p-1}{T^*}^{\frac{1}{2}-\frac{1}{2p}}+T^*\right) \leq m,$$

this implies that for all $t \in [0,T^*]$, $|z(t)|_{L^p(0,1)} \leq m$ and therefore $z \in \Sigma_p(m,T^*)$.

Step 2: G is a contraction on $\Sigma_p(m,T^*)$. Let us consider $v_1, v_2 \in \Sigma_p(m,T^*)$. We set $z = G(v_1 - v_2)$. Therefore,

$$z(t) = \int_0^t e^{(t-s)A}B(g(v_1+W_A) - g(v_2+W_A))ds.$$

Using Lemma 2.3, we get

$$\begin{aligned}|z(t)|_{L^p(0,1)} \leq C_1C_2(||\beta||_\infty+||\beta'||_\infty)\int_0^t \left((t-s)^{-\frac{1}{2}-\frac{1}{2p}}+1\right)&|g(v_1+W_A) \\ &- g(v_2+W_A)|_{L^p(0,1)}ds.\end{aligned}$$

Using the regularity of g, we obtain

$$\begin{aligned}|z(t)|_{L^p(0,1)} \leq & C_1C_2(||\beta||_\infty + ||\beta'||_\infty)||g'||_\infty \int_0^t \left((t-s)^{-\frac{1}{2}-\frac{1}{2p}}+1\right)|v_1-v_2|_{L^p(0,1)}ds \\ \leq & C_1C_2(||\beta||_\infty + ||\beta'||_\infty)||g'||_\infty \sup_{t\in[0,T^*]}\left(|v_1-v_2|_{L^p(0,1)}\right)\int_0^t \left((t-s)^{-\frac{1}{2}-\frac{1}{2p}}+1\right)ds \\ \leq & C_1C_2(||\beta||_\infty + ||\beta'||_\infty)\left(\frac{2p}{p-1}T^{*\frac{1}{2}-\frac{1}{2p}}+T^*\right)||g'||_\infty \sup_{t\in[0,T^*]}\left(|v_1-v_2|_{L^p(0,1)}\right).\end{aligned}$$

If we choose T^* small enough such that

$$C_1C_2(||\beta||_\infty + ||\beta'||_\infty)\left(\frac{2p}{p-1}T^{*\frac{1}{2}-\frac{1}{2p}}+T^*\right)||g'||_\infty < 1,$$

we obtain $|z(t)|_{L^p(0,1)} < ||v_1-v_2||_{\Sigma_p(m,T^*)}$. By taking the supremum in time on the left hand side of the inequality we obtain the desired result.

□

3.2 Global existence in time

Lemma 3.3. *Let $T\in(0,+\infty)$. If $v\in C([0,T];L^p(0,1))$ satisfies (2.5), then there exists two constants $\alpha_{p,\beta,g}$ and γ_p depending only on p, $||\beta||_\infty$, $||\beta'||_\infty$ and $||g||_\infty$, such that, for all $t\in[0,T]$, we have*

$$|v(t)|_{L^p(0,1)} \leq \left(|u_0|_{L^p(0,1)} + \frac{\alpha_{p,\beta,g}}{\gamma_p}\right).\exp\left(p\cdot\gamma_p\cdot t\right).$$

Proof. This proof is inspired by [2, Lemma 3.1], but for the readers convenience, we give the details.

We intend to work with regularised versions of the functions we considered before. For the initial condition, let $(u_0^n)_n$ be a sequence in $C^2(0,1)$ such that

$$u_0^n \to u_0 \text{ as } n\to+\infty \text{ in } L^p(0,1). \tag{3.3}$$

Let $W_A^n\in C^2([0,T]\times[0,1])$ be a sequence of regularised processes such that

$$W_A^n(t) = \int_0^t e^{(t-s)A}dW^n(s) \to W_A(t) \text{ in } C([0,T]\times[0,1]) \text{ as } n\to+\infty, \text{ a.s. } \omega\in\Omega. \tag{3.4}$$

Let v_n be the solution of

$$v_n(t) = e^{At}u_0^n + \int_0^t e^{(t-s)A}Bg_n(v_n+W_A^n)ds,$$

provided by Lemma 3.2, with $g_n\in C^2(\mathbb{R})$ a sequence of regularised functions such that

$$g_n\to g \text{ in } C(\mathbb{R}).$$

Given (3.3)-(3.4), the function v_n exists in a time interval $[0,T_n]$ with $T_n\to T^*$. Moreover, $v_n\in C^2([0,T]\times[0,1])$ and converges to v in $C([0,T^*],L^p(0,1))$ a.s. $\omega\in\Omega$. We also have that the regular function v_n satisfies

$$\frac{\partial v_n}{\partial t} - \frac{\partial^2 v_n}{\partial x^2} - \beta(x)\frac{\partial}{\partial x}g_n(v_n+W_A^n) = 0 \quad \text{a.s. } \omega\in\Omega. \tag{3.5}$$

Multiplying (3.5) by $|v_n|^{p-2}v_n$, and integrating over $[0,1]$, we obtain

$$\frac{1}{p}\frac{\partial |v_n|^p_{L^p(0,1)}}{\partial t} + (p-1)\int_0^1 |v_n|^{p-2}\left(\frac{\partial v_n}{\partial x}\right)^2 dx - \int_0^1 |v_n|^{p-2}v_n\beta(x)\frac{\partial}{\partial x}g_n(v_n+W_A^n)dx = 0. \tag{3.6}$$

Let us now consider the last term of the previous equation. We notice that it can be separated into two parts,

$$\int_0^1 |v_n|^{p-2}v_n\beta(x)\frac{\partial}{\partial x}g_n(v_n+W_A^n)dx = -\int_0^1 \beta'(x)g_n(v_n+W_A^n)|v_n|^{p-2}\cdot v_n dx \tag{3.7}$$

$$-(p-1)\int_0^1 \beta(x)g_n(v_n+W_A^n)|v_n|^{p-2}\frac{\partial v_n}{\partial x}dx. \tag{3.8}$$

Let us begin by considering (3.7),

$$\begin{aligned}\left|\int_0^1 \beta'(x)g_n(v_n+W_A^n)|v_n|^{p-2}\cdot v_n dx\right| \leq & ||\beta'||_\infty ||g_n||_\infty |v_n|^{p-1}_{L^{p-1}(0,1)} \\ \leq & \frac{1}{p}||\beta'||^p_\infty ||g_n||^p_\infty + \frac{p-1}{p}|v_n|^p_{L^{p-1}} \\ \leq & \frac{1}{p}||\beta'||^p_\infty ||g_n||^p_\infty + \tilde{C}_2\frac{p-1}{p}|v_n|^p_{L^p},\end{aligned} \tag{3.9}$$

where we have used for the first line the fact that β' and g are bounded, for the second line we simply have used Young inequality ($(ab \leq (1/p)a^p + b^{\frac{p}{p-1}}(p-1)/p)$ and finally for the third line we have used the Sobolev embedding theorem. Let us now consider (3.8),

$$\begin{aligned}&\left|(p-1)\int_0^1 \beta(x)g_n(v_n+W_A^n)|v_n|^{p-2}\frac{\partial v_n}{\partial x}dx\right| \\ \leq & (p-1)||\beta||_\infty ||g_n||_\infty \int_0^1 |v_n|^{p-2}\left|\frac{\partial v_n}{\partial x}\right| dx \\ \leq & (p-1)||\beta||_\infty ||g_n||_\infty \left(\int_0^1 |v_n|^{p-2}dx\right)^{\frac{1}{2}}\left(\int_0^1 |v_n|^{p-2}\left(\frac{\partial v_n}{\partial x}\right)^2 dx\right)^{\frac{1}{2}} \\ \leq & (p-1)\frac{||\beta||^2_\infty ||g_n||^2_\infty}{4}|v_n|^{p-2}_{L^{p-2}(0,1)} + (p-1)\int_0^1 |v_n|^{p-2}\left(\frac{\partial v_n}{\partial x}\right)^2 dx \\ \leq & \frac{p-1}{4}\left(\frac{2}{p}||\beta||^p_\infty ||g_n||^p_\infty + \frac{p-2}{p}|v_n|^p_{L^{p-2}(0,1)}\right) + (p-1)\int_0^1 |v_n|^{p-2}\left(\frac{\partial v_n}{\partial x}\right)^2 dx \\ \leq & \frac{p-1}{2p}||\beta||^p_\infty ||g_n||^p_\infty + \frac{(p-2)(p-1)}{4p}\tilde{C}_1|v_n|^p_{L^p(0,1)} + (p-1)\int_0^1 |v_n|^{p-2}\left(\frac{\partial v_n}{\partial x}\right)^2,\end{aligned} \tag{3.10}$$

where we have used for the second line the fact that β and g are bounded, for the third line we have used Hölder inequality with coefficients 2, for the fourth line Young inequality ($ab \leq a^2/4 + b^2$), for the fifth line we have used Young inequality ($ab \leq (2/p)a^{\frac{p}{2}} + b^{\frac{p}{p-2}}(p-2)/p$) and for the sixth line the Sobolev embedding theorem.

Injecting (3.9) and (3.10) into (3.6), we obtain

$$\frac{1}{p}\frac{\partial |v_n|^p_{L^p(0,1)}}{\partial t} \leq \left(\frac{||\beta'||^p_\infty ||g_n||^p_\infty}{p} + \frac{(p-1)||\beta||^p_\infty ||g_n||^p_\infty}{2p}\right) + \left(\tilde{C}_1\frac{(p-1)(p-2)}{4p} + \tilde{C}_2\frac{p-1}{p}\right)|v_n|^p_{L^p(0,1)}.$$

Thanks to Gronwall's lemma we obtain

$$|v_n|^p_{L^p(0,1)} \leq \left(|u_0^n|^p_{L^p(0,1)} + \frac{\alpha_{p,\beta,g_n}}{\gamma_p} \right) \cdot \exp\left(p \cdot \gamma_p \cdot t\right), \tag{3.11}$$

with

$$\alpha_{p,\beta,g_n} = \frac{||\beta'||^p_\infty ||g_n||^p_\infty}{p} + \frac{(p-1)||\beta||^p_\infty ||g_n||^p_\infty}{2p} \quad \text{and} \quad \gamma_p = \tilde{C}_1 \frac{(p-1)(p-2)}{4p} + \tilde{C}_2 \frac{p-1}{p}.$$

Passing to the limit as n goes to infinity we obtain the desired result.

□

Combining Lemma 3.2 and 3.3, we obtain Lemma 3.1 and it also implies the following result. If $\overline{\beta}$ is defined as in (A3), then Theorem 1.1 implies also the existence of a unique mild solution of the equation (1.8), which describes the homogeneous model.

Theorem 3.4 (Existence and uniqueness of a mild solution for (1.8))**.** *Assume (A1)-(A4). Let u_0 be given which is $\mathcal{F}_0-$measurable and such that $u_0 \in L^p(0,1)$ a.s. in Ω, for some $p \geq 3$. Let $T > 0$. Then there exists a unique mild solution of equation* (1.8)*, which belongs a.s. in Ω to $C([0,T], L^p(0,1))$.*

4 Homogenization

This section contains the proof of Theorem 1.3, which is a direct consequence of the following lemma, using (2.4) and the definition of $\overline{v}$.

Lemma 4.1. *Assume (A1)-(A2)-(A3). Let $T > 0$. Let $(v^\delta)_\delta$ be the sequence of mild solutions of* (2.5) *in $[0,T]$, provided by Lemma 3.1. Let $\overline{v}$ be the mild solution of* (2.7) *in $[0,T]$. Then*

$$v^\delta \to \overline{v} \text{ as } \delta \to 0 \quad \text{in } C([0,T], L^p(0,1)).$$

Proof of Lemma 4.1. The idea of the proof is to compare the two mild solutions

$$\overline{v}(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}\overline{\beta}(x)\frac{\partial}{\partial x}g(\overline{v} + W_A)ds, \tag{4.1}$$

and

$$v^\delta(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}\beta\left(\frac{x}{\delta}\right)\frac{\partial}{\partial x}g(v^\delta + W_A)ds. \tag{4.2}$$

Combining (4.1) and (4.2), we obtain that

$$\begin{aligned} |v^\delta - \overline{v}|_{L^p(0,1)} =& \left| \int_0^t e^{(t-s)A} \left[\beta^\delta(x) \frac{\partial}{\partial x} g(v^\delta + W_A) - \overline{\beta}(x) \frac{\partial}{\partial x} g(\overline{v} + W_A) \right] ds \right|_{L^p(0,1)} \\ \leq& \int_0^t \left| e^{(t-s)A} \frac{\partial}{\partial x} g(v^\delta + W_A) \left(\beta^\delta(x) - \overline{\beta}(x)\right) \right|_{L^p(0,1)} ds \qquad (4.3) \\ &+ \int_0^t \left| e^{(t-s)A}\overline{\beta}(x) \left[\frac{\partial}{\partial x} g(v^\delta + W_A) - \frac{\partial}{\partial x} g(\overline{v} + W_A) \right] \right|_{L^p(0,1)} ds. \qquad (4.4) \end{aligned}$$

Let us first treat the term inside the integral of (4.3). Using Lemma 2.3, we have that

$$\begin{aligned} &\left| e^{(t-s)A} \frac{\partial}{\partial x} g(v^\delta + W_A) \left(\beta^\delta(x) - \overline{\beta}(x)\right) \right|_{L^p(0,1)} \\ \leq& C_1 C_2 \left((t-s)^{-\frac{p+1}{2p}} + 1 \right) ||g||_\infty \left(|\beta^{\delta\prime} - \overline{\beta}'|_{L^p(0,1)} + |\beta^\delta - \overline{\beta}|_{L^p(0,1)} \right). \qquad (4.5) \end{aligned}$$

Let us now consider the term inside the integral of (4.4). Like before, using Lemma 2.3, we obtain

$$\begin{aligned}&\left|e^{(t-s)A}\overline{\beta}(x)\frac{\partial}{\partial x}\left[g(v^\delta+W_A)-g(\overline{v}+W_A)\right]\right|_{L^p(0,1)}\\ \leq & C_1C_2\left((t-s)^{-\frac{p+1}{2p}}+1\right)||\overline{\beta}||_\infty||g'||_\infty\left|v^\delta-\overline{v}\right|_{L^p(0,1)},\end{aligned}\tag{4.6}$$

Injecting (4.5) and (4.6) into (4.3)-(4.4), we obtain

$$\begin{aligned}|v^\delta-\overline{v}|_{L^p(0,1)}\leq & C_1C_2||g||_\infty\left(|\beta^{\delta\prime}-\overline{\beta}'|_{L^p(0,1)}+|\beta^\delta-\overline{\beta}|_{L^p(0,1)}\right)\int_0^t\left((t-s)^{-\frac{p+1}{2p}}+1\right)ds\\ &+C_1C_2||\overline{\beta}||_\infty||g'||_\infty\int_0^t\left((t-s)^{-\frac{p+1}{2p}}+1\right)|v^\delta-\overline{v}|_{L^p(0,1)}ds\\ \leq & C_1C_2||g||_\infty\left(|\beta^{\delta\prime}-\overline{\beta}'|_{L^p(0,1)}+|\beta^\delta-\overline{\beta}|_{L^p(0,1)}\right)\left(\frac{2p}{p-1}T^{\frac{1}{2}-\frac{1}{2p}}+T\right)\\ &+C_1C_2||\overline{\beta}||_\infty||g'||_\infty\int_0^t\left((t-s)^{-\frac{p+1}{2p}}+1\right)|v^\delta-\overline{v}|_{L^p(0,1)}ds\\ \leq & o_\delta(1)+C_1C_2||\overline{\beta}||_\infty||g'||_\infty\int_0^t\left((t-s)^{-\frac{p+1}{2p}}+1\right)|v^\delta-\overline{v}|_{L^p(0,1)}ds.\end{aligned}$$

Using Grownwall's lemma, we obtain

$$\begin{aligned}|v^\delta-\overline{v}|_{L^p(0,1)}\leq & o_\delta(1)\cdot\exp\left(2C_1C_2||\overline{\beta}||_\infty||g'||_\infty\int_0^t\left((t-s)^{-\frac{p+1}{2p}}+1\right)ds\right)\\ \leq & o_\delta(1)\cdot\exp\left(2C_1C_2||\overline{\beta}||_\infty||g'||_\infty\left(\frac{2p}{p-1}T^{\frac{1}{2}-\frac{1}{2p}}+T\right)\right)\\ \leq & o_\delta(1).\end{aligned}$$

Passing to the limit as δ goes to 0, we obtain our result.

□

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