PROPAGATION OF EXTREMELY SHORT PULSES IN NON-RESONANT MEDIA : THE TOTAL MAXWELL-DUFFING MODEL

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ABSTRACT

Propagation of extremely short pulses of electromagnetic field (electromagnetic spikes) is considered in the framework of the total Maxwell-Duffing model where anharmonic oscillators with cubic nonlinearities (Duffing model) represent the material medium and wave propagation is governed by the 1-d bidirectional Maxwell equations. This system of equations has a one parameter family of exact analytical solutions representing an electromagnetic spike propagating on a zero or a nonzero background. We find that the total

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Maxwell-Duffing equations can be written as a system in bilinear form and that the one-soliton solution of this system coincides with the steady state solution obtained previously.

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1 Introduction

Ultra short nonlinear pulses of an electromagnetic field, which contain as few as one half optical cycle, have recently attracted a great deal of attention [1]-[5]. The description of the evolution of the electromagnetic field was based on Maxwell equations or the subsequent wave equation. To describe a medium where electromagnetic waves propagate, one frequently uses an ensemble of oscillators or resonant atoms. If the oscillators are linear we obtain the important Lorentz model, which has been very useful to describe the propagation of an electromagnetic wave in a linear medium. The simplest generalization of the Lorentz model is obtained by adding an anharmonic term to the equation of the oscillator. This leads to the Duffing model in the case of a cubic nonlinearity [6, 7, 8]. In the framework of the Duffing model it was shown that solitary pulses of a unidirectional electromagnetic field ("electromagnetic bubbles") having an amplitude comparable to the atomic field and a duration down to $\sim 10^{-16}$ s may be expected. These bubbles propagate without dispersion as stable solitary waves. Very short pulses of this kind will be referred to as *extremely short pulses* (ESP).

In all these works, the oscillator represents the response of the high-frequency electronic degree of freedom to the electromagnetic field. The steady state ESP, which have been found heretofore, can be represented by a moving wave packet with a zero dominant frequency. Some authors call such pulses video pulses [9, 10] to emphasize the difference from the second kind of ESP with a non-zero dominant frequency. These (optical) ESP are considered in Ref. [5]. Hereafter we will consider ESP's of the first kind. Since the ESP spectrum is concentrated at low frequencies, ion oscillations may also give a considerable contribution to the full polarization of the medium. Nevertheless, we will neglect the ion component in the material response (the propagation of femtosecond pulses in a medium with a nonlinearity determined by both electronic and ionic (Raman-scattering) degrees of freedom was considered in Ref. [11]).

As the Duffing model is the simplest and most fruitful generalization of the Lorentz model, its investigation is very attractive. An objective of the present work is to study the propagation of linearly polarized ESPs in a nonlinear dispersive medium modeled by an anharmonic oscillator characterized by the cubic nonlinearities. The evolution of the electric field of ESP will be considered on the base of the Maxwell equations without any unidirectional reduction.

The paper is structured as follows. The model is derived in section 2 together with dynamical invariants or integrals of motion. Two families of moving ESP analytical solutions are given in section 3. In section 4 the bilinear form of these equations will be presented and we conclude in section 5.

2 Constitutive model

To consider the propagation of an extremely short electromagnetic pulse propagation in a nonlinear dispersive medium we need to use the total Maxwell wave equation and some model for the medium. Let us consider the standard Lagrangian density for an electromagnetic field taking into account the interaction with an ensemble of nonlinear oscillators:

$$\mathcal{L} = \frac{1}{2c_0^2}A_{,t}^2 - \frac{1}{2}A_{,z}^2 + 4\pi \sum_a \left\{ \frac{1}{2}mX_{a,t}^2 - \frac{1}{2}mX_a^2 - \frac{1}{4}\kappa_{3a}X_a^4 - \frac{e_*}{c_0}AX_{a,t} \right\} \quad (1)$$

Here we consider a plane electromagnetic wave, propagating along the z-axis and represented by the vector potential A. We use an anharmonic oscillators model (Duffing model) to reproduce the electronic response of an atom located at the spatial point indicated by the symbol a. The electrons are considered as particles in a potential well characterized by the displacements from their equilibrium position X_a. They oscillate with their own frequencies ω_a and are influenced by the electromagnetic field. In expression (1) e_* is the electric charge of the electron and κ_{3a} are anharmonicity coefficients. Hereafter, we will use m as a symbol for this effective mass, which accounts for the local Lorentz field effect. The partial derivatives we will denote as $\partial f/\partial t = f_{,t}, \partial f/\partial z = f_{,z}$ and so on.

The application of the variational procedure to the action related with the Lagrangian density (1) yields equations

$$A_{,zz} - c^{-2}A_{,tt} = (4\pi e_*/c_0)\sum_a X_{a,t},$$

$$mX_{a,tt} + m\omega_a^2 X_a + \kappa_{3a}X_a^3 = (e_*/c_0)A_{,t}$$

If one introduces the strength of the electric field $E = c_0^{-1} A_{,t}$, then the constituent equations of the model under consideration can be written as

$$E_{,zz} - c^{-2} E_{,tt} = (4\pi/c_0^2) P_{,tt} \tag{2}$$

$$X_{a,tt} + \omega_a^2 X_a + (\kappa_{3a}/m) X_a^3 = (e_*/m) E \tag{3}$$

The polarization P of the nonlinear medium is $P = \sum_a e_* X_a$.

Let us consider the case of a homogeneous broadening medium where all atoms have the same parameters, in particular $\omega_a = \omega_0$. Then we can write the polarization as $P = n_A e_* X$, where n_A is the density of the oscillators (atoms), and the index of the atom can be omitted.

We rescale the variables and fields as $\zeta = z\omega_0/c_0$, $\tau = \omega_0 t$ and $e = E/E_0$, $q = X/X_0$, where

$$E_0 = m\omega_0^2 X_0/e_* = m\omega_0^3 e_*^{-1} (2\mu/|\kappa_3|)^{1/2}, \quad X_0 = (2\mu m\omega_0^2/|\kappa_3|)^{1/2},$$

and we will also use the following parameters $\gamma = \omega_p/\omega_0$, $2\mu = \kappa_3 X_0^2/m\omega_0^2$, where $\omega_p = (4\pi n_A e_*^2/m)^{1/2}$ is the plasma frequency. In terms of the rescaled variables equations (2) and (3) take the form

$$e_{,\zeta\zeta} - e_{,\tau\tau} = \gamma q_{,\tau\tau} \quad , \quad q_{,\tau\tau} + q + 2\mu q^3 = \gamma e. \tag{4}$$

These equations will be named the *total Maxwell-Duffing equations* (or TMD-equations). To this system should be added the initial conditions:

$$e(\zeta = 0, \tau) = e_0(\tau), \quad e_{,\tau}(\zeta = 0, \tau) = e_0'(\tau),$$

and the boundary conditions

$$q(\zeta, \tau) = q_{,\tau}(\zeta, \tau) = 0, \quad \text{at} \;\; \tau \to \pm\infty.$$

related to the evolution of an initial electromagnetic solitary wave in a non-linear dispersive medium.

Note that the system of equations (4) can be rewritten in alternative form by introducing the new auxiliary field variable b by the relation $\partial e/\partial\zeta = \partial b/\partial\tau$. In this case the TMD-equations take the following form:

$$e_{,\zeta} = b_{,\tau} \quad , \quad b_{,\zeta} = e_{,\tau} + \gamma p \quad , \quad q_{,\tau} = p \quad , \quad p_{,\tau} + q + 2\mu q^3 = \gamma e. \tag{5}$$

It should be noted that the system of equations (4) can be derived as the Euler-Lagrange equations from the action functional

$$S = \int \mathcal{L}[q, a] d\tau d\zeta,$$

where now the new Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial a}{\partial \zeta}\right)^2 - \frac{1}{2}\left(\frac{\partial a}{\partial \tau}\right)^2 - \frac{1}{2}\left(\frac{\partial q}{\partial \tau}\right)^2 + \frac{1}{2}q^2 + \frac{\mu}{2}q^4 - \gamma q\frac{\partial a}{\partial \tau}. \tag{6}$$

Applying the variational procedure to the action S yields equations

$$\frac{\partial^2 a}{\partial \zeta^2} - \frac{\partial^2 a}{\partial \tau^2} = \gamma\frac{\partial q}{\partial \tau} \quad , \quad \frac{\partial^2 q}{\partial \tau^2} + q + 2\mu q^3 = \gamma\frac{\partial a}{\partial \tau} \tag{7}$$

Identifying a as a potential for the field e, so that $e = a_{,\tau}$, makes these equations identical to Eqs. (4).

From the Lagrangian density (6) we can get the density of moments of the fields a and q:

$$\pi_a(\zeta,\tau) = \frac{\partial \mathcal{L}}{\partial a_{,\zeta}} = a_{,\zeta}(\zeta,\tau) = b(\zeta,\tau) \quad , \quad \pi_q(\zeta,\tau) = \frac{\partial \mathcal{L}}{\partial q_{,\zeta}} = 0. \tag{8}$$

The second expression in (8) indicates that we have a degenerate Lagrangian, which leads to a constrained Hamiltonian system ($\pi_q(\zeta,\tau) = 0$ is a primary constraint) [12, 13].

Now one can get a canonical Hamiltonian density using the Legendre transform

$$\mathcal{H} = a_{,\zeta}\frac{\partial \mathcal{L}}{\partial a_{,\zeta}} - \mathcal{L} = \frac{1}{2}\left(a_{,\zeta}^2 + a_{,\tau}^2 + q_{,\tau}^2 - q^2 - \mu q^4 + 2\gamma q a_{,\tau}\right). \qquad (9)$$

The integration of this expression with respect to τ leads to the canonical Hamiltonian which is an integral of motion

$$H = \frac{1}{2}\int\limits_{-\infty}^{\infty}\left(e^2 + b^2 + q_{,\tau}^2 - q^2 - \mu q^4 + 2\gamma q e\right) d\tau. \qquad (10)$$

It is worth noting that there are two additional integrals of motion, which follow from the TMD-equations (5):

$$I_1 = \int\limits_{-\infty}^{\infty} e(\zeta,\tau) d\tau = \int\limits_{-\infty}^{\infty} a_{,\tau}(\zeta,\tau) d\tau = a(\zeta,\tau=\infty) - a(\zeta,\tau=-\infty), \qquad (11)$$

and

$$I_2 = \int\limits_{-\infty}^{\infty} b(\zeta,\tau) d\tau = \int\limits_{-\infty}^{\infty} \pi_a(\zeta,\tau) d\tau \qquad (12)$$

The magnitude of the first integral is defined by the boundary conditions only so that it can be interpreted as a topological charge in the Maxwell-Duffing model. The second integral is the canonical moment in this model.

3 Steady state solutions

Let us look for solutions of the TMD-equations as traveling waves with a non-varying profile. First consider solitary waves on a zero background, i.e. assume the following boundary conditions at $\tau \to \pm\infty$:

$$e(\zeta,\tau) = e_{,\tau}(\zeta,\tau) = 0 \quad , \quad q(\zeta,\tau) = q_{,\tau}(\zeta,\tau) = 0,$$

Substituting $e(\zeta,\tau) = e(\tau-\zeta/v)$, $\quad q(\zeta,\tau) = q(\tau-\zeta/v)$ into equation (4) and taking into account the boundary conditions one finds that the first equation of (4) results in $e = \alpha q/\gamma$, where $\alpha = \gamma^2 v^2/(1-v^2)$. The second equation of (4) can be transformed into the following ordinary differential equation in the variable $T = \tau - \zeta/v$

$$\frac{d^2 q}{dT^2} - (\alpha - 1)q + 2\mu q^3 = 0.$$

A non-singular solution of this equation exists only if $\alpha > 1$ and $\mu > 0$. In this case integrating results in the following expression [2, 7, 8]

$$q_{st}(\zeta,\tau) = \pm\sqrt{(\alpha-1)/\mu}\,\mathrm{sec\,h}\left[\sqrt{(\alpha-1)}\,(\tau-\zeta/v-\tau_0)\right]. \tag{13}$$

Here the integration constants τ_0 and α are the parameters of this steady state solitary wave. The strength of the electric field of the ESP is given by the following formula

$$e_{st}(\zeta,\tau) = \pm\alpha\gamma^{-1}\sqrt{(\alpha-1)/\mu}\,\mathrm{sec\,h}\left[\sqrt{(\alpha-1)}\,(\tau-\zeta/v-\tau_0)\right] \tag{14}$$

This solution describes the propagation of an electromagnetic spike with positive (+ sign) or negative (− sign) polarity. The condition for existence of this solution leads to the limitation of the velocity: $1 < v < (1+\gamma^2)^{-1/2}$.

Now let us consider solitary waves on a non-zero background, i.e. we assume the following boundary conditions at $\tau \to \pm\infty$::

$$e(\zeta,\tau) = e_0, \quad q(\zeta,\tau) = q_0 \quad , \quad e_{,\tau}(\zeta,\tau) = 0 \quad , \quad q_{,\tau}(\zeta,\tau) = 0, \tag{15}$$

$$\gamma e_0 = q_0 + 2\mu q_0^3 \tag{16}$$

Introduce the following fields $u(\zeta,\tau) = e(\zeta,\tau) - e_0$ and $f(\zeta,\tau) = q(\zeta,\tau) - q_0$. Instead of equations (4) we obtain the new system of equations

$$\frac{\partial^2 u}{\partial\zeta^2} - \frac{\partial^2 u}{\partial\tau^2} = \gamma\frac{\partial f}{\partial\tau} \quad , \quad \frac{\partial^2 f}{\partial\tau^2} + (1+6\mu q_0^2)f + 6\mu q_0 f^2 + 2\mu f^3 = \gamma\frac{\partial u}{\partial\tau}. \tag{17}$$

Under the assumption that $u(\zeta,\tau) = u(\tau - \zeta/v)$, $f(\zeta,\tau) = f(\tau - \zeta/v)$ the first equation leads to the relation $e = \alpha q/\gamma$, and the second equation takes the form

$$\frac{d^2 f}{dT^2} = (\alpha - 1 - 6\mu q_0^2) f - 6\mu q_0 f^2 - 2\mu f^3. \tag{18}$$

From (18) and the boundary conditions (15) it follows that

$$\left(\frac{df}{dT}\right)^2 = (\alpha - 1 - 6\mu q_0^2) f^2 - 4\mu q_0 f^3 - \mu f^4. \tag{19}$$

A non-singular solution of this equation exists when $\alpha_1 \equiv \alpha - 6\mu q_0^2 > 1$ and $\mu > 0$. This condition leads to the limitation of the velocity of the ESP

$$1 < v^2 < v_c^2 = \frac{1 + 6\mu q_0^2}{1 + \gamma^2 + 6\mu q_0^2}.$$

The solution of equation (19) can be written as

$$f^{(\pm)}(\zeta,\tau) = \frac{\pm(\alpha_1 - 1)}{\sqrt{(2\mu q_0)^2 + \mu(\alpha_1 - 1)^2}\cosh[(\alpha_1 - 1)^{1/2}(\tau - \zeta/v - \tau_0)] \pm 2\mu q_0} \tag{20}$$

The steady state pulse of electromagnetic wave propagating on a constant electric background is then given by

$$e^{(\pm)}(\zeta,\tau) = e_0 \pm \frac{\alpha(\alpha_1 - 1)}{\sqrt{(2\mu q_0)^2 + \mu(\alpha_1 - 1)^2}\cosh[(\alpha_1 - 1)^{1/2}(\tau - \zeta/v - \tau_0)] \pm 2\mu q_0} \tag{21}$$

Unlike the case of ESP propagation without background, here ESPs of different polarities have different amplitudes.

4 Bilinear form of the total Maxwell-Duffing equations

If the following substitutions

$$e = \frac{g}{h} \quad , \quad b = \frac{c}{h} \quad , \quad q = \frac{f}{h} \tag{22}$$

are used, then equations (5) can be rewritten as

$$\begin{aligned}
\frac{1}{h^2} D_\zeta (g \cdot h) - \frac{1}{h^2} D_\tau (c \cdot h) = 0, \\
\frac{1}{h^2} D_\zeta (c \cdot h) - \frac{1}{h^2} D_\tau (g \cdot h) = \gamma \frac{1}{h^2} D_\tau (f \cdot h), \\
\frac{1}{h^2} D_\tau^2 (f \cdot h) - \frac{f}{h^3} D_\tau^2 (h \cdot h) + \frac{f}{h} + 2\mu \frac{f^3}{h^3} = \gamma \frac{g}{h},
\end{aligned} \tag{23}$$

where the Hirota D-operators $D_\zeta(a \cdot b) = a_{,\zeta} b - a b_{,\zeta}$, $D_\tau(a \cdot b) = a_{,\tau} b - a b_{,\tau}$ have been introduced [14]. To derive the last equation we follow the rule

$$\frac{\partial^2}{\partial \tau^2} \left(\frac{f}{h} \right) = \frac{1}{h^2} D_\tau^2 (f \cdot h) - \frac{f}{h^3} D_\tau^2 (h \cdot h).$$

Multiplying the first equation by h^2, the second equation one by h^3, the third one by h^2 and collecting the terms of order h^{-1}, we can write the resulting equations as a system of bilinear ones

$$\begin{aligned}
D_\zeta (g \cdot h) &= D_\tau (c \cdot h), \\
D_\zeta (c \cdot h) &= D_\tau (g \cdot h) + \gamma D_\tau (f \cdot h), \\
D_\tau^2 (f \cdot h) &= (\gamma g - f) h, \\
D_\tau^2 (h \cdot h) &= 2\mu f^2
\end{aligned} \tag{24}$$

We will use the usual method [14, 15] to solve equations (24) by writing

$$g = g_1 e^{\theta}, \quad c = c_1 e^{\theta}, \quad f = f_1 e^{\theta}, \quad h = 1 + h_1 e^{\theta} + h_2 e^{2\theta}, \tag{25}$$

where $\theta = k\zeta - \Omega\tau$. Substituting this into (24) results in the equations:

$$\begin{aligned}
kg_1e^{\theta} - kh_2g_1e^{3\theta} + \Omega c_1e^{\theta} - \Omega h_2c_1e^{3\theta} &= 0, \\
kc_1e^{\theta} - kh_2c_1e^{3\theta} + \Omega g_1e^{\theta} - \Omega h_2g_1e^{3\theta} + \Omega\gamma f_1e^{\theta} - \Omega\gamma h_2f_1e^{3\theta} &= 0, \\
\Omega^2 f_1e^{\theta} + \Omega^2 h_2f_1e^{3\theta} - (\gamma g_1 - f_1)(e^{\theta} + h_1e^{2\theta} + h_2e^{3\theta}) &= 0, \\
\Omega^2 h_1e^{\theta} + 4\Omega^2 h_2e^{2\theta} + \Omega^2 h_2h_1e^{3\theta} - \mu f_1^2e^{2\theta} &= 0.
\end{aligned}$$

Equating the coefficients of the different powers of e^{θ} to zero, one obtains the system of equations that define $c_1, f_1, g_1, h_{1,}h_2$:

$$\begin{aligned}
kg_1 + \Omega c_1 &= 0 \quad , \quad kc_1 + \Omega(g_1 + \gamma f_1) = 0, \\
\Omega^2 f_1 - (\gamma g_1 - f_1) &= 0 \quad , \quad 4\Omega^2 h_2 = \mu f_1^2 \quad , \quad h_1 = 0.
\end{aligned}$$

From these relations one can get

$$\begin{aligned}
h_1 &= 0 \quad , \quad h_2 = \mu f_1^2 (2\Omega)^{-2}, \\
c_1 &= -\gamma^{-1}(k/\Omega)(1 + \Omega^2) f_1 \quad , \quad g_1 = \gamma^{-1}(1 + \Omega^2) f_1,
\end{aligned}$$

and the "dispersion relation"

$$k^2 = \frac{\Omega^2(1 + \gamma^2 + \Omega^2)}{(1 + \Omega^2)}. \tag{26}$$

It should be pointed out that in the low frequency limit this formula yields $k^2 = \Omega^2(1 + \gamma^2)$, while in the high-frequency limit it yields $k^2 = \Omega^2$.

Thus, we found a one-soliton solution of the bilinear equations (24) which can be written as

$$\begin{aligned} g &= \gamma^{-1}(1+\Omega^2) f_1 e^{\theta}, \\ c &= -\gamma^{-1}(k/\Omega)(1+\Omega^2) f_1 e^{\theta}, \\ f &= f_1 e^{\theta} \quad , \quad h = 1 + \mu f_1^2 (2\Omega)^{-2} e^{2\theta}. \end{aligned}$$

These relations yield a solution of the TMD-equations (6) which is consistent with the steady state solution obtained earlier:

$$e_{sol}(\zeta,\tau) = \frac{\gamma^{-1}(1+\Omega^2) f_1 e^{\theta}}{1+\mu(f/2\Omega)^2 e^{2\theta}}, \tag{27}$$

$$b_{sol}(\zeta,\tau) = -\frac{\gamma^{-1}(k/\Omega)(1+\Omega^2) f_1 e^{\theta}}{1+\mu(f/2\Omega)^2 e^{2\theta}}, \tag{28}$$

$$q_{sol}(\zeta,\tau) = \frac{f_1 e^{\theta}}{1+\mu(f/2\Omega)^2 e^{2\theta}}.$$

If we introduce a new constant of integration $\exp\theta_0 = \pm\mu^{1/2} f_1/2\Omega$, then the one-soliton expression for the normalized electric field of ESP can be written as

$$e_{sol}(\zeta,\tau) = \pm\left(\frac{1+\Omega^2}{\mu^{1/2}\gamma}\right)\frac{\Omega}{\cosh(\theta+\theta_0)}. \tag{29}$$

The velocity of this soliton is defined as $v = \Omega/k$. Taking into account the dispersion relation (26) one can obtain

$$v^2 = \frac{1+\Omega^2}{1+\gamma^2+\Omega^2}.$$

The magnitude of this velocity lies in the interval $(1, (1+\gamma^2)^{-1/2})$. This is the same result as for a steady state pulse obtained above. Using the expression for velocity we can see that $\alpha = 1+\Omega^2$, and the expression for the steady state pulse (14) coincides with expression (29).

5 Conclusion

We have introduced and analyzed a model for the propagation of extremely short unipolar pulses of an electromagnetic field in a medium represented by anharmonic oscillators with a cubic nonlinearity. The model under consideration takes into account the dispersion properties of both linear and nonlinear responses of the medium. This model is the simplest generalization of the well known Lorentz model used to describe linear optical properties in condensed matter. The cubic nonlinearity is the first type of anharmonic correction to the Lorentz model and it results in the Duffing oscillator. Here we consider the total Maxwell-Duffing model in detail. The Lagrangian picture of the TMD model was considered and three integrals of motion were found. Two families of exact analytical solutions, with positive and negative polarities, have been found as moving solitary pulses. The first kind of steady state ESP is an electromagnetic spike propagating in a nonlinear medium. It was discussed early in [2, 7, 8, 16]. A new kind of steady state ESP is an electromagnetic spike propagating on a non-zero electric background. There are both bright and dark ESP. Unlike the ESP on a zero background, here pulses of different polarities have different amplitudes.

We found that the TMD equations can be represented in bilinear Hirota's form. In the case of a zero background the one-soliton solution of the bilinear equations was obtained. It coincides with the expression of a steady state ESP. There are many examples when the bilinear form of the nonlinear evolution equations leads to the existence of two-soliton solutions without ensuring complete integrability or/and the existence of N-soliton solutions. However in this particular case we were not able to obtain a two-soliton solution of the TMD equations.

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